Ph. D. Thesis on

D-branes, gauge/string duality and noncommutative theories

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1. J. Gomis, K. Kamimura and T. Mateos, “Gauge and BRST generators for space-time non-commutative U(1) theory,” JHEP **0103** (2001) 010 [arXiv:hep-th/0009158].

2. T. Mateos and A. Moreno, “A note on unitarity of non-relativistic non-commutative theories,” Phys. Rev. D **64** (2001) 047703 [arXiv:hep-th/0104167].

3. J. Gomis and T. Mateos, “D6 branes wrapping Kaehler four-cycles,” Phys. Lett. B **524** (2002) 170 [arXiv:hep-th/0108080].

4. J. Brugues, J. Gomis, T. Mateos and T. Ramirez, “Supergravity duals of noncommutative wrapped D6 branes and supersymmetry without supersymmetry,” JHEP **0210** (2002) 016 [arXiv:hep-th/0207091].

5. T. Mateos, J. M. Pons and P. Talavera, “Supergravity dual of non-commutative N = 1 SYM,” Nucl. Phys. B **651** (2003) 291 [arXiv:hep-th/0209150].

6. J. Brugues, J. Gomis, T. Mateos and T. Ramirez, “Commutative and noncommutative N = 2 SYM in 2+1 from wrapped D6-branes,” Class. Quant. Grav. **20** (2003) S441 [arXiv:hep-th/0212179].

7. J. Gomis, T. Mateos, P. J. Silva and A. Van Proeyen, “Supertubes in reduced holonomy manifolds,” Class. Quant. Grav. **20** (2003) 3113 [arXiv:hep-th/0304210].

8. D. Mateos, T. Mateos and P. K. Townsend, “Supersymmetry of tensionless rotating strings in $AdS_5 \times S^5$, and nearly-BPS operators,” JHEP **0312** (2003) 017 [arXiv:hep-th/0309114].

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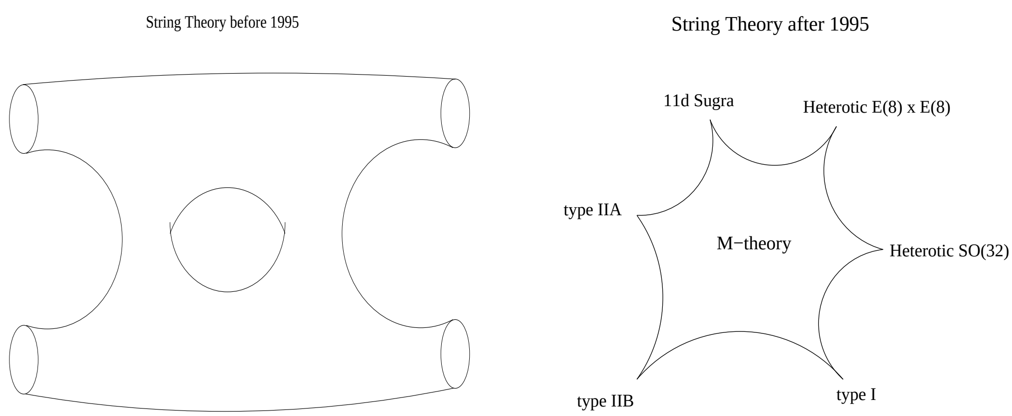
CONTENTS

I. INTRODUCTION

It has been almost ten years since the discovery of D-branes [1], and it is fair to say that nothing has been the same anymore. Polchinski used to start his talks with a transparency like this,



which encodes the deep transformation of our view of String Theory that took place during 1995-1997. This thesis deals about D-branes and some of the main new lines of research that they opened.

Polchinski's diagram refers to the development of a series of dualities that allowed to relate the five 10d superstring theories which are known to be free from anomalies. Some of these dualities mapped the strong coupling regime of one of the string theories to the weak coupling of another one. This is the example of the type IIB S-duality, a case in which the original and final theories are the same. For cases like these, D-branes turned out to be fundamental, as they provided the non-perturbative states needed to complete the net of connections between the various Hilbert spaces that we observe at the vertices of the M-theory diagram.

Dualities were one of the first developments in which D-branes played a crucial role, and they are probably among the most important achievements towards the understanding of what String Theory really is. There was however another key property of D-branes that was just waiting for its exploitation, a property that turned String Theory into one of the most multidisciplinary fields of physics: the fact that at low energies they can be described by ordinary gauge field theories[1]. If the spatial dimension of the D-brane is p, in which case we talk about a Dp-brane, one is led to consider gauge theories in $p+1$ dimensions. Progressively, most of the field theory phenomena that we were familiar with acquired a geometrical interpretation in terms of how a particular setup of D-branes is embedded in a certain 10d manifold. The list of examples of this reinterpretation is uncountable.[2] Let us just mention some of the most intuitive ones:

- scalars in the field theory are reinterpreted as giving the embedding of the D-brane in its transverse space; they are actually the Goldstone bosons corresponding to the background symmetries broken by the presence of the D-brane,
- the R-symmetry group is reinterpreted as the group of isometries in the D-brane transverse space,
- (the breaking of) the Poincaré group of the field theory corresponds to (the breaking of) the Poincaré isometry along the D-brane directions,
- many field theory instantons and monopoles can be interpreted as various strings/branes ending or intersecting D-branes.

But in order for a reinterpretation to be useful and not just philosophy, it must be able to provide new results. It turns out that the *gauge theory phenomena* $\leftrightarrow$ *D-brane* relation has been extremely fruitful; it has allowed to find new results in gauge theories based on stringy intuition and, conversely, to find new results in D-brane physics based on a purely field-theoretical approach. This thesis contains examples of results in both directions, the most sophisticated one possibly being the way that D-branes implement the field-theory phenomenon known as *twisting*, to be discussed later.

[1] To be strict, this is not always the case as sometimes there are massive excitations that cannot be decoupled.

[2] A basic review of some of these is given in chapter II.

I.1 AdS/CFT

Apart from carrying gauge theories on their worldvolumes, there is another crucial property of D-branes: they are charged under the Ramond-Ramond (RR) field potentials. This allowed for an identification of D-branes with the supergravity solitons with nonzero RR-potentials that had been known for some years [2]. Despite being solutions of the low energy effective action of the various string theories, they had been waiting for an interpretation; they could not describe the backreaction of any state in the perturbative Hilbert space of string theory, as none of them couples minimally to the RR fields. D-branes came to fill this gap bringing supergravity back to the game. Somehow, there was a transition between the description of D-branes as 2d conformal field theories with boundaries and their description in terms of supergravity solutions. For example, by putting more and more branes on top of each other, the gravitational scale of the system starts growing, the D-branes become 'fat' and the backreaction cannot be neglected. People started to realize that these two points of view could be made functional. Some observables corresponding to the low energy gauge theory on the D-branes started to be computable from the supergravity side.

It was Maldacena [3] who finally made the conjecture[3] that, at least in the case of N D3-branes, its low energy $\mathcal{N}=4$ $SU(N)$ superconformal Yang-Mills was dual to type IIB string theory in the near horizon region of its supergravity solution, $AdS_5 \times S^5$. This provided the first concrete example of the conjecture made decades earlier by 't Hooft that non-abelian gauge theories could be described in terms of string theories at least at large N. Maldacena's duality was even more surprising as the particular string dual of the 4d gauge theory was actually a string theory in *ten dimensions.* Somehow, the degrees of freedom of the type IIB strings had to be 'holographically projected' to its boundary, which is conformal to 4d Minkowski space. This made the ideas of holography, a discipline that had been proposed independently of string theory, enter the game as well.

Maldacena's conjecture, which is also referred to as AdS/CFT duality, had better be of a weak/strong nature, as we do not see anything like closed strings or branes in a perturbative analysis of SYM. That this is so made the conjecture both powerful and difficult test, not to mention 'prove'. In principle, it allowed to compute the same observable in both sides (although using different languages) and say that the results do not necessarily match, as they correspond to opposite regimes of either the field

[3] This conjecture is motivated and explained in detail in chapter III.

or the string theory. However, this encounters a huge obstacle from its very beginning: how do we translate the degrees of freedom from one theory to the other and confirm that we are computing the same observable? The answer to this question is not straightforward at all and it is fair to say that the states of both Hilbert spaces that have been able to be mapped are of zero measure compared to the number of total states. To construct this dictionary it is good to have some symmetries at hand, like the bosonic group of global symmetries of the SYM theory, $SO(2,4) \times SO(6)$ or its full supersymmetry group $PSU(2,2|4)$. These symmetries should be present for all values of the coupling, thus they should be visible in the closed string side; they indeed correspond to the isometries of the $AdS_5 \times S^5$ background. Classifying states in irreps of these groups helps building the dictionary as we will repeatedly see during this thesis.

But even if we had the complete dictionary, the aim of testing the duality would remain almost unreachable if all that we can use is perturbative SYM and supergravity. The latter is the only approximation that we can deal with in the string side as the IIB sigma model in $AdS_5 \times S^5$ is, to date, not possible to quantize. There are however a series of observables that do allow for a comparison, those whose values are known not to depend on the coupling. These observables are typically related to BPS states in supergravity and BPS operators in the SCFT. The energy of the former and the conformal dimension of the latter are completely fixed by the underlying superconformal algebra, which relates them to the other charges that they may have. Consider the example of the operators

$$\mathcal{O} = \mathrm{Tr}\left(X^J\right) , \qquad X \equiv \phi_1 + i\phi_2 \,, \tag{I.1}$$

where $\phi_{1,2}$ are two of the scalars of the $\mathcal{N} = 4$ supermultiplet. These operators are invariant under half of the Poincaré supersymmetries and a straightforward argument based on the $PSU(2,2|4)$ superalgebra[4] shows that their conformal dimension Δ must be $\Delta = J$. These operators are dual to supergravity excitations with angular momenta J along the S^5, which are also 1/2-BPS and their energy is $E = J$.

The impossibility of testing the AdS/CFT duality beyond BPS-protected quantities was enormously improved in the work of Berenstein, Maldacena and Nastase (BMN) [4]. They showed that the Penrose limit of $AdS_5 \times S^5$ along a null geodesic in the S^5 was dual to a subset of the $\mathcal{N} = 4$ operators with large charge J under a $U(1)$ subgroup of the $SO(6)$ R-symmetry.

[4] See section III.4.3 for a prove of this and similar but more general relations, and the appendix A for a detailed discussion on BPS operators.

This Penrose limit leads to the only maximally supersymmetric IIB background [5] that remained to be exploited: a certain class of pp-wave. The simplicity of this background allows for a quantization of the string σ-model, and this opens a huge region in the parameter space of the two theories where both are simultaneously accessible with our present techniques. Let us remark that the operators that survive the BMN limit are, despite being non-BPS, very close to those in (I.1); they have a number of insertions of other fields which is small compared to J.

Soon after the work of BMN, a shortcut was provided by Gubser, Klebanov and Polyakov in a paper [6] where they proposed that some σ-model solitons in $AdS_5 \times S^5$ were able to provide similar answers within a classical approximation, bypassing the need to quantize in RR backgrounds. Their ideas were immediately applied to solitons in many other complicated backgrounds which are believed to have a gauge theory dual, yielding a number of predictions for the strong coupling behavior of some of their observables.

A qualitatively new set of results started with the papers of Frolov and Tseytlin [7, 8] where they considered σ-model solitons that carried three angular momenta (J_1, J_2, J_3) along the S^5. The novelty was that they were able to match exactly [9, 10, 11, 12, 13, 14, 15] the classical energy $E(J_i)$ of the solitons to a one-loop computation of the conformal dimension of the operators

$$\mathcal{O} = \mathrm{Tr}\left(X^{J_1} Y^{J_2} Z^{J_3}\right) \; + \text{ permutations}, \tag{I.2}$$

with $Y = \phi_3 + i\phi_4$ and $Y = \phi_5 + i\phi_6$, by interpreting the one-loop anomalous dimension matrix as an integrable spin-chain Hamiltonian [16, 17]. These operators, and their corresponding string theory states, are very far from the 1/2-BPS BMN operators, so it looks like supersymmetry has nothing to do with these tests. If this was so, then there are a number of immediate difficulties and still open questions that progress along these lines is having to face. The first problem is that many of the solitons that provide successful comparisons were shown to be unstable. The second problem is that one has to justify why are quantum σ-model corrections negligible against the classical result. We will see that this is a very involved problem that has only been answered to one loop and for a particular class of solutions[5]. We will investigate this subject in great detail in section III.4 and we will give arguments why this particular correspondence is being so successful. Some recent results [18] seem to support our proposal as we discuss in the conclusions section. A deeper understanding is, however, still required.

[5] Note that some of these quantum corrections were performed about an unstable vacuum!

The AdS/CFT duality and more generally the relation between the two open/closed descriptions of D-branes has also provided new ways to look at a problem. Consider a set of N Dp-branes and M Dq-branes in flat space. The probe picture is consistent at weak coupling as long as both N and M are small, so that the backreaction can be neglected. One can compute the interaction among them by the standard techniques and then conclude whether they attract, repel or do not feel the others' presence at all. The leading order of this interaction typically involve a one-loop diagram of open strings or, equivalently, a tree level diagram of closed strings with sources (boundary states).

The open/closed string description comes into the game when we let N grow. At some point, the Dp-branes description is more adequate in terms of their supergravity solution. If in this process we kept M fixed, we end up with M Dq-probes in the background of N Dp-branes. We will explore a wide set configurations in which the original setup is such that the final M Dq-probes are embedded as an $AdS \times \Sigma$ submanifold in the near horizon region of the N Dp-branes background, with Σ a compact submanifold. The standard case is a setup of D3/D5 branes as the following array indicates

IIB	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
D3	–	–	–	–						
D5	–	–	–		–	–	–			

We will see in section III.5 that substituting the D3's by their $AdS_5 \times S^5$ background can lead to an embedding of the D5's as an $AdS_4 \times S^2$ submanifold in which the S^2 has maximal volume within the S^5. Note that there is no topological obstruction for the S^2 to collapse to a point in the S^5. However, the embedding must be stable as the original set of branes is known to be 1/4- supersymmetric. The apparent paradox can be resolved by noting that the tachyonic instabilities of the S^2 embedding have masses above the Breitenlohner-Friedman bound [19, 20, 21] from the point of view of the field theory in the AdS_4 factor.

We will study many other examples and show that the type of interaction between D- or M-branes can be understood in terms of tachyonic masses being above or below the corresponding BF bound. This analysis will also lead us to the possibility of introducing non-supersymmetric but stable D-branes in $AdS \times S$ backgrounds. According to the AdS/dCFT duality (where d stands for *defect*), the probes correspond to the addition of matter multiplets in the dual CFT; these are confined to live in the submanifold where the probe intersects de AdS boundary and hence the name

of *defect*. Despite being still work in progress, we will present candidates for such stable but non-supersymmetric embeddings in which the D-/M-brane probes are AdS-filling. This should correspond to the addition of non-supersymmetric matter in the dual theory without any confining restriction, *i.e.* ordinary matter. We cannot be conclusive at this stage yet about the stability of these embeddings, but we hope to report on it in the near future.

I.2 Beyond AdS/CFT: the gauge/string duality

The possibility of having a strong coupling dual of a theory like QCD motivated a lot of effort in trying to extend the AdS/CFT duality to field theories other than the $\mathcal{N}=4$. Any such extension has finally earned the name of *gauge/string duality*, reserving 'AdS/CFT' for those cases in which the field theory involved is conformal. It was clear that the original picture of N D3-branes in flat 10d space had to be made more sophisticated if one wanted to end with less than 16 supercharges. Some attempts were initially based on replacing the S^5 background by cosets S^5/Γ or by cones over other Einstein 5d manifolds. Some other attempts introduced small perturbations to the $\mathcal{N}=4$ Lagrangian, which are dual to deformations of the AdS_5 that do not change its asymptotics. We comment on these in section IV.3.

We will mostly consider a different approach in which the flatness of both the ambient space and of the brane's worldvolume is completely abandoned. The preservation of some fraction of supersymmetry by the background will lead us to the concept of special holonomy manifolds, whereas the preservation by the embedded worldvolume will lead us to the concept of calibrated cycles. We will see that the particular way in which the worldvolume gauge theory of the brane manages to preserve supersymmetry had actually been discovered 15 years ago by Witten [22]. By means of a mechanism called *twisting*, one is able to put supersymmetric field theories in some curved backgrounds. The number of preserved supersymmetries turns out to be less than the corresponding theory in flat space, which is precisely what we were looking for.

This field theory intuition is crucial in order to build the closed string duals of these less than maximally supersymmetric theories, and it constitutes one of the most sophisticated examples of the interplay between gauge theories and D-branes that we mentioned above. This will help us to construct the closed string dual of an $\mathcal{N}=2$ $SU(N)$ SYM theory in 2+1 dimensions without any matter other than the vector multiplet. We will

analyze what string theory can tell us about its moduli space and discuss that it is tentative to interpret it as an all-loops resummation.

Indeed, because the closed string dual is constructed with D6-branes, the uplift of this solution to 11d supergravity will produce an explicit metric for an eight-dimensional Calabi-Yau space [23]. Indeed, it is through this D-brane intuition that so many metrics with special holonomy have been built in the recent years. Whereas for Calabi-Yau spaces we have Yau's theorem guaranteing the existence of a unique Ricci flat metric with $SU(n)$ holonomy in each Kähler class, there is no such theorem for $Spin(7)$ and G_2 manifolds. By wrapping D-branes it has been able to prove the existence of some of such metrics by simply constructing them. Thus starting from the twist of field theories we have ended with a purely mathematical progress!

Field theory intuition	→	String Th. intuition	→	Maths result
Susy field theories can be put in curved spaces by twisting them	→	D6-branes can be wrapped in special holonomy manifolds	→	Explicit construction of metrics with special holonomy

There are a couple of features of the duals that extend the AdS/CFT correspondence that must be stressed. The first one is technical and refers to the fact that what ultimately simplified the construction of the supergravity solutions was the use of gauged supergravities, as proposed by Maldacena and Núñez [24]. These are much simpler than the IIA/IIB/11d supergravities as they arise after a truncation of an infinite number of modes. Furthermore, D-branes arise as domain-wall solutions of them, a fact that dramatically simplifies the ansatz. We will see during this thesis that, unfortunately, gauged supergravities cannot always be used. The second point is actually a drawback common to most of the AdS/CFT extensions achieved until now. It turns out that in the limit in which supergravity is valid, and we recall that this is the only possibility due to the incapability of quantizing the corresponding σ-models, the dual field theory is not just what one was looking for at the beginning but it contains an infinite number of undecoupled degrees of freedom. For example, when the field theory involved is a deformation of the $\mathcal{N}=4$ SYM with operators of typical mass M, the supergravity approximation is valid only when M is of the same order as the dynamically generated scale Λ_{QCD}; thus the confining or strong coupling phase does not correspond to the QCD-like theory alone.

In the examples of wrapped branes, one expects to recover an ordinary SYM theory in the non-compact part of the D-brane when the volume of

the cycle that they wrap tends to zero, *i.e.* in the IR. However, such small cycles typically imply that the background curvature is larger than the string scale, which renders the supergravity approximation invalid. Insisting on the use of supergravity means that the physics on the non-compact part of the brane contains an infinite set of undecoupled Kaluza-Klein modes.

Everything we have mentioned in this section is expanded and discussed in detail in chapter IV.

I.3 Noncommutative theories in string theory

Having exploited D-branes to obtain AdS/CFT-like dualities, let us change subject and analyze another branch of String Theory that D-branes allowed to open. Whereas the quantization of the string σ-model in flat space is rather straightforward as it is essentially gaussian, it becomes a difficult problem as soon the background becomes more involved. Finding quantizable backgrounds is an important task, as some of them can lead to a better understanding of string theory in different regions of its moduli; we already saw above the great relevance of the quantization in the IIB pp-wave background.

In [25], quantization of the open string σ-model with D-brane boundary conditions in a background with a constant NS-NS B_2-field was achieved, and this led to some new surprises: the low energy limit turned out to be described by noncommutative (NC) gauge theories. To be more specific, only those with magnetic or light-like B_2-fields admit a consistent field theory limit, whereas those with electric ones do not admit a decoupling of all the string massive modes[6]. The NC actions are obtained from the usual commutative ones by replacing the standard product of functions by a $*$-product defined as

$$(F * G)(x) = F(x) \exp\left[\frac{i}{2}\theta^{\mu\nu}\left(\overleftarrow{\partial_{x^\mu}}\overrightarrow{\partial_{x^\nu}} - \overleftarrow{\partial_{x^\nu}}\overrightarrow{\partial_{x^\mu}}\right)\right] G(x)\,, \qquad \text{(I.3)}$$

where $\theta^{\mu\nu}$ measures the intensity of the noncommutativity between space-time coordinates,

$$x^\mu * x^\nu - x^\nu * x^\mu = i\theta^{\mu\nu}\,. \qquad \text{(I.4)}$$

The uncertainty principle states then that an attempt to localize a wave-function in one direction makes it increasingly delocalized in another one.

[6] We will see however in section V.2.5 that in electric backgrounds there is a different limit that leads to a theory in which open strings decouple from closed string.

Maybe a better way to understand it is that given two functions f and g with support in a region of small size Δ, the $*$-product $h = f * g$ is supported in a region of size θ/Δ. The extreme example is the $*$-product of two delta functions, which gives a constant function with infinite support.

This property is behind one of the most intriguing aspects of NC theories, an aspect which arises at the quantum level when trying to compute loop corrections to observables. It turns out that the IR and the UV physics of the theory are completely undecoupled, a property that frontally jeopardizes the Wilsonian approach to renormalization[7]. Perhaps the simplest example is a particular diagram that contributes to the 1-loop self energy of a NC ϕ^4 theory as

$$\Gamma^2(p) = \frac{\lambda}{96\pi^2}\left[\Lambda_{\text{eff}}^2 - m^2 \ln\left(\frac{\Lambda_{\text{eff}}^2}{m^2}\right)\right] , \tag{I.5}$$

where

$$\Lambda_{\text{eff}}^2 = \frac{1}{1/\Lambda^2 + p \circ p} , \tag{I.6}$$

Λ is a UV hard cutoff, and $p \circ p = -p^\mu \theta^2_{\mu\nu} p^\nu$. This result seems to be finite if we just send the cutoff to infinity, *i.e.* if we include arbitrarily high energy modes; however, this leaves us with an IR divergence as $p \to 0$. Similarly, the contribution is then divergent as $\theta \to 0$, which means that the commutative limit of the quantum NC theory is not the commutative quantum theory. These IR divergences would not be present if we kept Λ finite, which suggests that they are actually caused by modes in the UV.

This phenomenon, known as UV/IR mixing, motivated a critic examination of these theories. Were they actually sensible at all? It was soon found that those NC theories that did arise as consistent field theory limits of string theory inherited its unitarity [27] and causality [28], providing stronger evidence that they are solid quantum field theories on their own, and that the UV/IR mixing required more study.

Chapter V deals with NC theories at the classical and the quantum level. We will study the unitarity at one-loop of some NC scalar field theories and confirm that it is not violated unless electric components of the B-field are turned on. We will also examine a possible way in which the non-decoupling of the stringy modes in these cases can be traced into the lack of unitarity. In particular, we will try to restore it by enlarging the asymptotic Hilbert space of the field theory (adding the so-called χ-particles). We will see that this is specially difficult in a toy model of a non-relativistic NC ϕ^4 theory in 2+1 dimensions.

[7] See [26] for a recent PhD thesis on this subject.

The enormous qualitative difference between magnetic and electric backgrounds can be understood by the appearance of non-locality involving time in the latter. The quantization of theories which are non-local in time is not straightforward at all and most approaches consider their Lagrangians as a function of a field and (at best) all its time derivatives. In chapter VII we will thoroughly discuss a more solid method for constructing a Hamiltonian formalism for time non-local theories which is based on the original idea of Llosa and Vives [29], further developed in [30]. We will then apply it to settle a consistent Hamiltonian and BRST formalism for a NC $U(1)$ gauge theory in four dimensions in which the notions of conserved charges and symmetry generators appear naturally. We remark that our analysis does not apply only to NC theories, but to any theory which is non-local in time. In particular, it has been recently applied to the study of tachyon condensation within the p-adic string and String Field Theory [31].

I.4 Linking NC theories, AdS/CFT and gauge/string duality

The suspicion that the UV/IR mixing of NC theories may be an artifact of the Feynman diagrammatic expansion is just one of the motivations to study them by alternative methods[8]. If dual closed string backgrounds could be found, they could shed some new light to this phenomenon and provide new non-perturbative information.

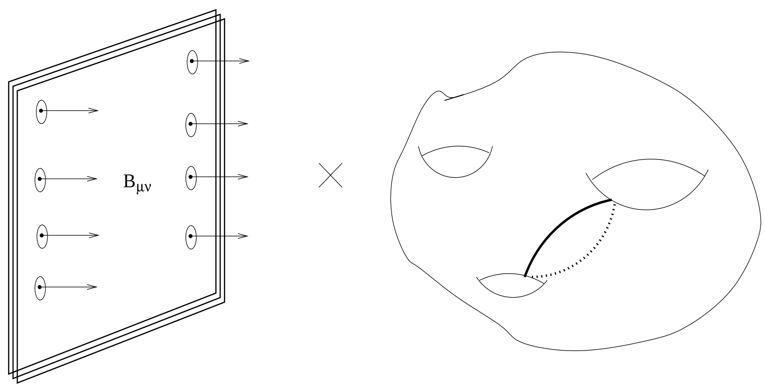
The first duals were constructed by Maldacena and Russo [32], and Hashimoto and Itzhaki [33]. In particular they were able to study the magnetic NC deformation of the superconformal $\mathcal{N} = 4$ in four dimensions. Some expected features of noncommutativity were visible in this background; in particular, UV/IR mixing seems to slightly modify the geometry in the IR but it completely disappears in the deep infrared. The disadvantage of the $\mathcal{N} = 4$ in this respect is that it is absent of UV divergences in its perturbative diagrams. This implies that the UV/IR phenomenon does not seem to be present, at least in perturbation theory, which would perfectly fit with the prediction from the supergravity dual[9].

Thus we are led to the enterprize of finding supergravity solutions of NC

[8] We have in mind now only the magnetic cases which do arise in string theory.

[9] The issue of whether the IR or the commutative limit of the $\mathcal{N} = 4$ is actually smooth requires a more careful study. Similarly, it is not always the case that the planar limit of a NC theory corresponds to the commutative one. See [34] for a good discussion on this issues.

theories with less than maximal supersymmetry. Fortunately, we found that the same ideas that allowed for a reduction of supersymmetry by wrapping D-branes in special holonomy manifolds extend to backgrounds in which we turn on magnetic B-fields. The incompatibility is avoided as long as the B-field flux along the special holonomy manifold vanishes. This is not an impediment for our purposes, as in the IR we want to end up with a NC field theory on the flat noncompact part of the D-brane. The schematic picture is as follows,



We will be able to construct the supergravity duals of two NC theories[10]:

- a $U(N)$ NC $\mathcal{N}=1$ SYM in 3+1 (section VI.3),
- a $U(N)$ NC $\mathcal{N}=2$ SYM in 2+1 (section VI.4).

For the first theory we discuss a good amount of nonperturbative properties derived from the closed string dual: the presence of UV/IR mixing, confinement, the β-function and chiral-symmetry breaking. We will see an interesting property which is absent from the commutative counterpart: the new scale introduced by the noncommutativity can be fine-tuned so that it allows for a decoupling of the KK modes. We advance here that such a decoupling can only be achieved by setting the NC scale to be the largest one in the problem, thus it does not allow to end up with a 'realistic' theory anyway. In constructing the dual of the second theory, we will encounter the unexpected problem that the phenomenon known as 'supersymmetry without supersymmetry' affects NC theories in a completely different manner

[10] Note that we always consider $U(N)$ instead of $SU(N)$ gauge groups when dealing with NC theories. This is because, unlike in commutative theories, the $U(1)$ photon couples to rest of fields in the gauge multiple, see page 160.

to their commutative counterparts. We will prove that gauged supergravities are useless in constructing NC backgrounds and provide an alternative method which involves a series of T-dualities.

I.5 Map of the thesis

We have introduced the three main subjects in which the whole work of this thesis is embedded:

1. D-branes,
2. AdS/CFT duality and its extension to less supersymmetric theories,
3. NC field theories.

Let us sketch what how the original work is distributed along the thesis.

- Chapter II includes the part of the work that deals purely with D-branes, in particular with the possibility of constructing supertubes [35] in a large class of curved manifolds [36]. The chapter includes the introductory material to D-branes that will be needed in the rest of the thesis.

- Chapter III contains the part of the work that deals purely with the AdS/CFT. We study the possibility of testing the duality beyond supergravity and supersymmetry, as reported in [37, 38]. We include unpublished work in collaboration with D. Mateos and P.K. Townsend about the various possibilities of embedding D-brane probes in $AdS \times \Sigma$ submanifolds of 10d and 11d $AdS \times S$ backgrounds. We will see that the Breitenlohner-Freedman bound of the field theory that lives in the AdS factor of the D-brane is able to tell us whether the various involved D-branes attract, repel or do not feel any force at all.

- Chapter IV includes the part of the work that deals purely with the extension of the AdS/CFT to less supersymmetric field theories. After introducing all the necessary concepts with some detail, we describe the construction of the supergravity dual of an $SU(N)$ $\mathcal{N}=2$ SYM theory in three dimensions. Its 11d description provides a metric for an 8d noncompact Calabi-Yau manifold. We analyze the its moduli space from the supergravity side, based on the results reported in [39, 40].

- Chapter V contains the part of the work that deals noncommutative *field* theories. After a brief review of how they where introduced in string theory, we analyze some of its main quantum properties. In particular, as reported in [41], we study the unitarity of a non-relativistic NC ϕ^4 theory and the possibility of adding χ-particles to restore unitarity in the electric case.

- Chapter VI includes the part of the work devoted to link the extensions of AdS/CFT via wrapped branes with the NC theories. It is based on the results reported in [42, 43, 40].

- Finally, chapter VII contains the work devoted to settle a Hamiltonian and BRST formalism for any non-local field theory in time, such as the electric NC theories described above. It is based on the results of [44].

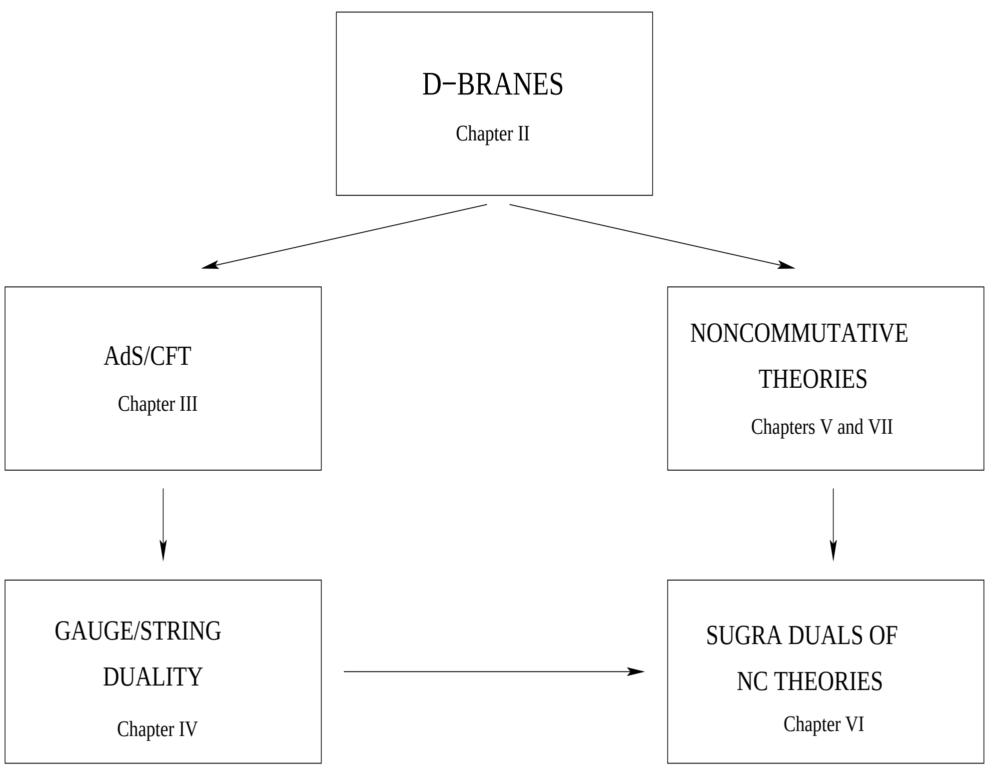


Fig. I.1: The thesis at a glance.

II. PHYSICS OF D-BRANES

This chapter covers some basics of the physics of D-branes, giving a special emphasis to those topics that will be needed in the thesis. We will mainly concentrate on the conceptual issues, trying to build a self-consistent base for the three topics of the next chapters: NC theories, AdS/CFT correspondence and gauge/string duality. There are excellent reviews in the literature (*e.g.* [45, 46]) and we refer the reader to them for technical details and extra material.

After this short review, and as part of the work during this thesis concerning only D-brane physics, we introduce the supertubes in section II.3. As will be shown, supertubes intensively exploit the sophisticated dynamics of D-branes. We extend the original construction of [35, 47] and show that supertubes can be constructed supersymmetrically in a huge variety of curved spaces enabling, among other things, the construction of their closed strings description in terms of IIB supergravity backgrounds preserving from 1/4 to 1/32 supersymmetries.

II.1 Perturbative definition and spectrum of a single D-brane

In string perturbation theory, Dp-branes are defined as (p+1)-dimensional hypersurfaces (let us call them Σ_{p+1}) where open strings are allowed to end. Their dynamics are therefore described by the excitations of open strings with a mixed set of boundary conditions

$$
\begin{array}{rll}
\text{Neumann BC's along } \Sigma_{p+1}: & \partial_n X^\mu = 0\,, & \mu = 0, ..., p\,, \\
\text{Dirichlet BC's normal to } \Sigma_{p+1}: & \partial_t X^i = 0\,, & i = p+1, ..., (D-1)\,,
\end{array}
$$

where ∂_n and ∂_t stand for normal and tangent derivatives to the surface swept by the string worldsheet in a D-dimensional spacetime.

Having such a perturbative definition, let us ask what the spectrum of the open strings subject to D-brane boundary conditions is. Whereas D-

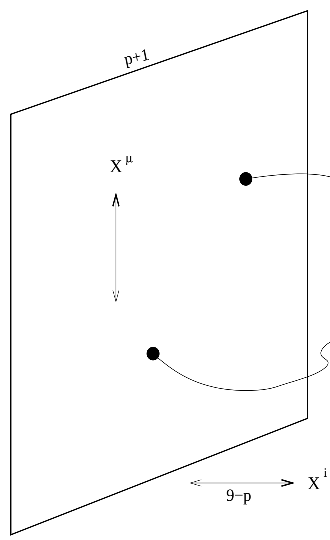


Fig. II.1: Definition of D-branes in perturbative open string theory

branes exist in all string theories containing open strings, the detailed spectrum of their fluctuations varies. All along this thesis, we will be mainly concerned with branes in IIA and IIB superstring theories. To describe their spectra, it is worth noticing first that their branes can preserve, at most, 16 supersymmetries. This is easy to understand since the left/right worldsheet supercharges of type II theories are separately conserved in closed topologies, but they must be identified in open topologies due to the boundary conditions. As a result only one half of the background supercharges can be preserved (at most), and this is the case of *e.g.* flat D-branes in Minkowsky space. This is one of the simplest and best understood brane configurations in string theory, so let us take a pause to study them a bit further.

II.1.1 Low energy effective action for a single D-brane

We start reviewing the spectrum of open superstrings in 10d Minkowsky space $\mathcal{M}_{10}$ with Neumann BC's for all of the scalar fields, which is identified with a D9-brane. We still have the option to choose NS or R BC's for the fermion fields on the worldsheet. The vacuum in the NS sector is a tachyon, which is GSO-projected out; the next physical states form a vector representation of $SO(8)$, and therefore provide us with the on-shell degrees of freedom of a massless spacetime abelian vector field A_μ. On the other hand the vacuum in the R sector is, after the GSO projection, just a

Majorana-Weyl spinor representation of $SO(8)$. As expected by the argument above, this is precisely the content of the unique $\mathcal{N}=1$ vector multiplet in ten dimensions. All other massive modes in each sector have masses of the order $1/l_s$, and they form independent supersymmetric representations at each mass level. Finally, the spectrum of any other flat Dp-brane in $\mathcal{M}_{10}$ can be found by dimensionally reducing the just mentioned 10d spectrum to p+1 dimensions. It is very important to realize that various polarizations of the vector A_μ are transformed into scalar degrees of freedom for lower-dimensional branes. The expectation values of these scalars can be interpreted as parameterizing the position of the brane in its transverse space; indeed, they are precisely the massless Goldstone bosons associated to the breaking of the global background Poincaré symmetry by the presence of the hyperplane. This interpretation can be supported in a number of different ways, as we will keep encountering in the rest of this thesis.

Summarizing, the massless spectrum of open strings ending on a single D-brane consists of a $U(1)$ gauge multiplet with 16 supercharges in (p+1) dimensions and the rest of excitations have spacetime masses of order $1/l_s$. At low energies ($E \ll 1/l_s$) only the massless excitations remain relevant, and the originally stringy theory reduces to a field theory governed by the usual (p+1)-dimensional Super Yang-Mills (SYM) actions for the mentioned gauge multiplet. This is typically proven either by analyzing the low energy limit of the various S-matrix processes or by imposing the vanishing of the Weyl anomaly at first order in l_s^2.

Indeed one can do better than just writing the SYM action for these massless modes. Was one to take into account the interactions of the massless modes with the rest of massive string modes, the SYM action would then be just the first term of the complete action, thought of as an expansion in l_s^2. It has been possible to resum all such contributions for the case of constant gauge fields, and the complete bosonic action is of a Dirac-Born-Infield (DBI) type

$$S_{DBI} = -\mu_p \int_{\Sigma_{p+1}} d^{p+1}\sigma\, e^{-\Phi} \left(\sqrt{-\det\left(P\left[G+B\right] + 2\pi l_s^2 F\right)}\right) , \qquad \text{(II.1)}$$

with

$$\mu_p = \frac{1}{(2\pi)^p l_s^{p+1}} , \qquad \text{(II.2)}$$

and the operator P denoting the pullback of spacetime fields to the worldvolume.

Before completing the discussion of the single-brane effective actions, we need to take into account that branes must act as sources of closed

strings, as can easily be seen just by worldsheet duality. In particular, branes *can* gravitate and emit dilaton and RR fields quanta. When the reaction of the background can be neglected, we say that we are in the probe approximation; its validity depends only on the scale of energies we want to study. We will later see that one can take limits where this approximation is never valid (*e.g.* by considering an infinite number of branes on top of each other), and the backreaction must then be taken into account. Indeed the action (II.1) already includes the coupling to the background metric and dilaton; it was not until [1] that the coupling to the RR fields was discovered. The DBI action must then be supplemented with these new couplings, which turn out to be of a Wess-Zumino type

$$S_{WZ} = \mu_p \int_{\Sigma_{p+1}} P\left[\bigoplus_n C_n e^B\right] e^{2\pi l_s^2 F}\,, \tag{II.3}$$

where C_n are the various RR n-form fields, and we have written the action as a formal sum of forms of different degree; the integration only picks up those with the correct degree to be integrated over Σ_{p+1}.

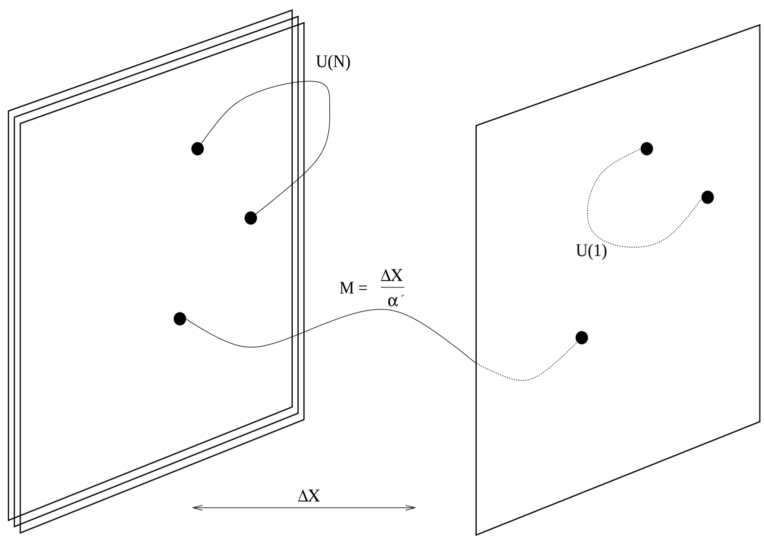
II.1.2 Multiple D-branes

An interesting phenomenon occurs when we consider N D-branes in flat space which, by definition, is equivalent to adding Chan-Paton factors to the endpoints of the open strings. To begin with, one may worry about whether considering this situation is worth at all. It could well happen that no such a *static* configuration is achievable because both branes attract or repel; this is what one would expect for objects that gravitate and couple to gauge fields. Indeed, most part of this thesis is build over exceptions to this naive expectation.

Let us leave for chapter IV the general discussion and concentrate again on branes in flat space. It is not hard to see that if all the branes are parallel, then the set of boundary conditions still preserve the same supersymmetries as a single brane. Being a supersymmetric configuration, it minimizes the energy and it is therefore stable. The physical explanation for this is that the interactions among the D-branes due to interchange of closed string modes is exactly zero at each mass level. For example, at the massless level, the gravitational and dilatonic attraction is cancelled by the repulsion due to RR-fields exchange.

What about the spectrum? Quantisation leads now to a massless spectrum that consists of a $U(1)^N$ gauge supermultiplet. These massless modes

correspond to the low energy excitations of the open strings with both endpoints in only one D-brane. Quite remarkably, the next massive states have masses which are now not just proportional to $1/l_s$ but to $\Delta X/l_s$ times $1/l_s$. These are the lowest energy excitations of strings with endpoints on different branes, ΔX being the distance between them. The states with these masses have precisely the right quantum numbers to be interpreted as the W-bosons for spontaneous symmetry breaking of $U(N)$ to $U(1)^N$. It is then understandable that when any two branes are placed on top of each other the W-bosons become massless, and the gauge symmetry is enhanced from $U(1) \times U(1) \to U(2)$. Putting all of the D-branes together just provides



us with a $U(N)$ supermultiplet in $(p+1)$ dimensions with 16 supercharges, obtainable again from the ten dimensional one by dimensional reduction. Note that this includes a set of transverse scalar fields which, being in the same multiplet as the gauge fields, transform in the adjoint of the gauge group.

What about the dynamics now? Repeating the same arguments above, one finds that the low energy effective action for the massless fields is governed by the non-abelian $U(N)$ SYM action in $(p+1)$ with 16 supersymmetries obtainable by dimensional reduction from the ten dimensional one. An important feature to note for later reference is that the resulting YM coupling in terms of the string parameters is

$$g_{YM}^2 = (2\pi)^{p-2} g_s l_s^{p-3} \,. \tag{II.4}$$

Another important result is that the action contains a positive-definite po-

tential for the scalar fields ϕ^i of the type

$$V \sim \sum_{i,j} [\phi^i, \phi^j]^2 \,. \tag{II.5}$$

Here we meet again a geometrical interpretation of a usual field theory phenomena. There exists a moduli space of vacua which minimize (II.5) parametrized by all the vev of the scalars lying in the Cartan subalgebra of the gauge group. They are therefore simultaneously diagonalizable and their eigenvalues can be interpreted again as parametrizing the positions of the N branes in their transverse space. Higgsing the gauge group just corresponds to giving a vev to one of this scalars, and therefore to moving one of the branes apart. As mentioned above, the open strings with one endpoint in the stack and the other in the fugitive brane become the massive W-bosons.

We finish this review of the physics of multiple D-branes by mentioning that the non-abelian generalization of the DBI action is still not known completely, and it is not even clear that such an enterprize makes sense at all. However, some extra terms are well-known and specially the ones involving the RR fields have recently attracted a lot of attention, since they are capable to provide couplings of low-dimensional branes to RR fields sourced by higher dimensional ones. We refer the reader to [48] for a recent review on this subject.

II.1.3 $\mathcal{N} = 4$ SYM

It is worth illustrating the previous general discussion in a particular example. As this will be one of the most important cases, let us consider in more detail the configuration of N D3 branes in a flat IIB background. The low energy theory is a $U(N)$ $\mathcal{N} = 4$ SYM in 3+1 dimensions. It turns out that a field theory action with such properties is uniquely determined by the coupling constant g_{YM} and the rank of the gauge group N. The field content is: one gluon, 6 scalars and 4 Majorana gluinos. We will not need to consider the fermions for most part of this thesis, but for the purposes of writing the action in a simple way, we will group the 4 gluinos into a 10d 16-component Majorana-Weyl spinor. The notation for the fields is then

$$A_\mu(x), \qquad \phi_i(x), \quad i = 1, \ldots, 6 \qquad \chi_\alpha(x), \quad \alpha = 1, \ldots 16 \,, \tag{II.6}$$

where all fields are valued in the adjoint of the gauge group. The action is

$$S = \tfrac{2}{g_{YM}^2} \int d^4x \, \mathrm{Tr} \Big\{ \frac{1}{4} (F_{\mu\nu})^2 + \frac{1}{2} (D_\mu \phi_i)^2 - \frac{1}{4} [\phi_i, \phi_j] [\phi_i, \phi_j] + \frac{1}{2} \bar{\chi} \not{D} \chi - \frac{i}{2} \bar{\chi} \, \Gamma_i \, [\phi_i, \chi] \Big\} , \tag{II.7}$$

where the Γ-matrices are the 10d ones.

Although this action can be built just by imposing the mentioned properties, it turns our that it enjoys a good extra amount of symmetry: superconformal invariance. The super-Poincaré generators $\{P^\mu, M_{\mu\nu}, Q_\alpha\}$ and the R-symmetry ones T^A are accompanied by the generators of special conformal transformations K^μ, dilations D and conformal supersymmetries S_α The whole symmetry group is $PSU(2,2|4)$, whose bosonic subgroup is $SO(2,4) \times SU(4)_R$. Under the $SU(4)_R$ R-symmetry A_μ is a singlet, the 4 fermions are in the fundamental, and the scalars transform as a vector of the homeomorphic group $SO(6)$. We will discuss in detail this superalgebra and its representations in chapter 3 and in the appendix A.

The everyday case is that the classical scale invariance of the action is immediately broken at the quantum level, which is actually a virtue rather than a problem. For example, this allows massless QCD to give an approximate description of the real world, where we observe everything but scale invariance. There is typically no way to make sense of the UV divergences of a QFT without introducing a scale. The present $\mathcal{N} = 4$ SYM theory is an exception since, at least in perturbation theory, no single correlation function exhibits UV divergences. Even instanton contributions are finite, and the theory is believed to be UV finite. As a consequence, the β-function is exactly zero and the superconformal group remains as a symmetry of the quantum theory.

Although N and g_{YM} completely determine the action, they do not uniquely determine the theory; one still needs to specify the vacuum in which he wants to live. Unlike most non-supersymmetric theories, supersymmetric ones are often unable to dynamically determine the lowest energy state. Indeed, they typically have a continuous of such vacua (called moduli space) parametrized by the vacuum expectation value (vev) of some fields. For our case, if we assume that the vevs of the fermions and the gauge field are zero, we can give a set of different vevs to the scalars such that they all minimize the potential

$$V \sim [\phi_i, \phi_j] \, [\phi_i, \phi_j] \, . \tag{II.8}$$

We then speak of different phases or branches of the moduli space. Being positive definite, minimization of (II.8) is just the equation $V = 0$, and this

has two types of solutions:

- *The superconformal phase* is characterized by all vevs of the scalars being zero. This clearly preserves the whole superconformal group.

- In *the Coulomb phase*, one has a nonzero vev for at least one of the scalars. Note that if there are more than one such nonzero vevs, we must require them to be in the Cartan subalgebra of $SU(N)$, so that these fields commute and we still have $V = 0$. Having r such nonzero vevs will spontaneously break $SU(N) \to U(1)^r \times SU(N-r)$. As soon as this happens, we will observe at large distances the appearance of massless photons with their usual Coulomb interactions, and hence the name for this branch. Needless to say, conformal invariance is also broken due to the scales introduced by the scalars.

We saw in section II.1.2 that the scalars have the interpretation of giving the position of the D-branes in their transverse space. Therefore, the Coulomb phase is associated to having one (or more) branes separated from the others. The degrees of freedom of the strings connecting separated branes became the W-bosons, with masses of the order

$$M_W \sim \frac{\Delta X}{l_s^2} \,. \tag{II.9}$$

II.2 D-branes as solutions of closed string theory

The picture we wish to present here is similar to the more familiar picture of electrons and electric fields. Consider putting an electron in an almost empty space (with weak background fields). We would start describing the electron by its worldline relativistic action plus a minimal coupling

$$S = \int_{worldine} [ds + A_1] \,. \tag{II.10}$$

If the space had been truly empty before putting the electron, we would have not considered the second term. This is the analog of the probe approximation we were using in the previous section. On the other hand, because of its coupling to the photon fields, we are also used to describing the electron by the electric field that it produces, which at long distances behaves as $V \sim -e/r$. It this picture, the electron is a delta-function source for the potential

$$\vec{\nabla}^2 V \sim e \cdot \delta^{(3)}(\vec{x} - \vec{x}_0) \,. \tag{II.11}$$

This second point of view is the one we want to adopt here, *i.e.* the description of D-branes as closed string backgrounds with (typically) δ sources. As the low energy dynamics of closed strings is supergravity the problem of singularities is a little more sophisticated than in (II.11), and one has to deal with horizons, proper assymptotics, causal structures, naked singularities...

So, let us remain in type IIA/IIB string theory and look for solutions of their corresponding supergravities describing our D-branes. Our aim here is the most modest one: we want solutions that describe *flat D-branes in flat Minkowsky space*. We would like to stress that

- *solutions corresponding to flat D-branes in flat space are not flat*, just like the electric field derived from (II.11) is not zero, and it describes an electron in an empty space.
- *the solutions may even not contain any D-brane!* The criterium of whether *there is a brane or not* in a supergravity solution is normally answered by whether it is a solution of the supergravity equations of motion *with or without* source[1]. According to this, the background (II.11) contains an electron since it does not solve the Maxwell equations alone but

$$S \sim \int d^4x \, (F_{\mu\nu}F^{\mu\nu} + j^\mu A_\mu) \, , \qquad j^\mu \sim \delta^{(3)}(\vec{x} - \vec{x}_0)\delta_0^\mu \, . \qquad \text{(II.12)}$$

  A typical stringy counter example to this is the geometry of the D3-brane solution. It is absolutely regular everywhere and it solves the vacuum (without extra sources) IIB supergravity equations of motion.

Especially in cases where the second point applies, it is standard to use the expression *geometric transition*; the object we started with disappears and only a curved geometry with fluxes remains.

The solutions we are looking for were found in 1991 [2], much earlier than the understanding of Dp-branes in open string theory; they had been called simply p-branes, and the name is still used to emphasize the supergravity picture we are describing. They can be found by requiring that they have the properties expected for a D-brane:

1. The background must involve only those massless fields that couple to the D-brane, *i.e.* the graviton, the dilaton and a $p + 1$ RR potential.

[1] See however [49] for a clear discussion of the different concepts of charge used in the literature.

2. It must have the isometries $ISO(1,p) \times SO(9-p)$ corresponding Poincaré invariance in the worldvolume and rotational invariance in the transverse space.

3. It must break preserve 16 supersymmetries.

The solutions to these requirements are

$$\begin{aligned} ds^2_{10} &= H^{-\frac{1}{2}} dx^2_{0,p} + H^{\frac{1}{2}} dx^2_{p+1,9}, & \text{(II.13)} \\ e^{\phi} &= g_s H^{\frac{3-p}{4}}, & \text{(II.14)} \\ A_{p+1} &= -\frac{1}{2}(H^{-1}-1) dx^0 \wedge ... \wedge dx^p\,, & \text{(II.15)} \end{aligned}$$

where

$$\begin{aligned} H &= 1 + \frac{R^{7-p}}{r^{7-p}}\,, \qquad r^2 = (x^{p+1})^2 + ... + (x^9)^2\,, & \text{(II.16)} \\ R^{7-p} &= d_p\, g^2_{YM} N \alpha'^{5-p}\,, \qquad d_p = 2^{7-2p} \pi^{\frac{9-3p}{2}} \Gamma\left(\frac{7-p}{2}\right)\,. & \text{(II.17)} \end{aligned}$$

We are using an optimized notation in which

$$dx^2_{0,p} \equiv -(dx^0)^2 + (dx^1)^2 + ... + (dx^p)^2\,. \qquad \text{(II.18)}$$

Let us remark that these solutions are adapted to describing a single stack of N p-branes. The function H can be any harmonic function on the transverse space, and the configuration still solves the e.o.m. and preserves the same number of supersymmetries. So the obvious way to describe multiple stacks of parallel p-branes is to choose

$$H = 1 + \sum_i \frac{R_i^{7-p}}{|\vec{r} - \vec{r}_i|^{7-p}}\,, \qquad R_i^{7-p} = d_p\, g^2_{YM} N_i \alpha'^{5-p}\,. \qquad \text{(II.19)}$$

In general these metrics present a null curvature singularity at $r = 0$. This is the case of all p-branes with $p \neq 3, 6$. For $p = 6$ the singularity is time-like and for $p = 3$ there is no singularity at all (it is a coordinate singularity) and one can analytically continue the solution inside the horizon [50].

Validity of the solutions:
It is extremely important to take care of the regimes of validity of the description just given. A careful case-by-case analysis was given in [51]. In this thesis we will be more interested in the near-horizon limits rather than the full solutions themselves. Therefore we postpone this discussion until chapter III, after the introduction of the AdS/CFT ideas.

II.3 An example of brane dynamics: supertubes

Having discussed how D-branes appear in string theory, we are ready to start exploiting the two descriptions that they admit.

II.3.1 Generalities of D-brane stabilization

All the examples we have seen so far described flat D-brane configurations in flat space. We saw that these are completely stable configurations that preserve a high amount of supersymmetry. It is enough to deal with such configurations for many purposes, *e.g.* to motivate the AdS/CFT correspondence. Many other purposes, however, require the consideration of more complicated configurations in less trivial backgrounds. The gauge/string correspondence and the appearance of NC gauge theories are examples of this. One can think of different ways of complicating the picture, like

1. considering non-flat D-branes,
2. putting them in non-flat backgrounds,
3. intersecting D-branes of (possibly) different dimensions.

All three issues have been intensively studied in the literature and they have led to many interesting insights in different areas of physics. A general problem which is common to the 3 generalizations is how to stabilize a given D-brane configuration. Being extended massive and charged objects, different points interact among each other and with the background, and stability is an exceptional situation rather than a standard one.

There are cases in which supersymmetry guarantees stability because supersymmetric states typically have the minimum possible energy for their given charges. This statement is powerful because it is normally proven by algebraic methods, thus they do not depend on the perturbative approximations that are normally needed to be made. We will see in detail how these arguments work in chapter IV. Note however that supersymmetry is not always necessary, as there exist examples of stable but non-supersymmetric brane configurations (see *e.g.* [52, 53]).

Let us guess which are the best candidates for being stable but non-trivial D-brane configurations. We start by keeping the background space flat and trying to curve the D-branes. As soon as we move away from the flat hyper-plane case, the D-brane tension will create a tendency to modify such an embedding; indeed, if a part of the D-brane is compact,

such a tendency will be towards its collapse. Maybe the simplest option is to change the background for a topologically non-trivial one, so that the collapse is prevented because the brane wraps a non-contractible cycle. This will be the subject of section IV.5.

In this chapter we will concentrate on supertubes, which have the distinctive property that they provide D-brane stabilization in *flat space*. Despite the difficulties mentioned above, such configurations are possible precisely because the dynamics of D-branes are much richer than those of a simple relativistic extended object.

II.3.2 Preliminaries for the construction of the supertube in the open string picture

The purpose of this section is to provide the background and the intuition needed to understand why supertubes were possible to be constructed. Their generalizations to curved backgrounds is also heuristically motivated. We postpone a more formal treatment to the following sections.

In order to achieve the construction of a curved brane in flat space we will exploit various couplings that appear in the low energy dynamics of the open strings attached to the brane. Recall that the action consists of two pieces II.1-II.3

$$\begin{aligned} S &= S_{DBI} + S_{WZ}\,, \\ S_{DBI} &= -\mu_p \int_{\Sigma_{p+1}} d^{p+1}\sigma\, e^{-\Phi} \sqrt{-\det\left(P\left[G+B\right] + 2\pi l_s^2 F\right)}\,, \qquad \text{(II.20)} \\ S_{WZ} &= \mu_p \int_{\Sigma_{p+1}} P\left[\bigoplus_n C_n e^B\right] e^{2\pi l_s^2 F}\,, \qquad \text{(II.21)} \end{aligned}$$

and that it is exact for constant worldvolume gauge fields F_2.

The first point we want to make is that the electric and magnetic fields in F_2 are sources for background D(p-2)-branes and fundamental strings F1, respectively. The reason for the former is that the presence of a magnetic F_2 flux in the worldvolume of the Dp-brane couples to the RR-potential of a D(p-2) through one of the terms in the WZ action (II.21) as

$$S_{WZ}|_{C_{p-1}} \sim \int_{\Sigma_{p+1}} F_2 \wedge C_{p-1} \sim q_{p-2} \int_{\Sigma_{p-1}} C_{p-1}\,, \qquad \text{(II.22)}$$

where q_s is the flux of the magnetic field on a spatial 2d submanifold of the brane's worldvolume. The latter is due to the coupling of the electric

components of F_2 to the electric components of the background B_2-field, which is the source for F1. This coupling appears when expanding the DBI action (II.20)

$$S_{DBI} \ni \int_{\Sigma_{p+1}} F_{\mu\nu} B^{\mu\nu} \sim \int_{\Sigma_{p+1}} *F_2 \wedge B_2 \sim q_{F1} \int_{\Sigma_2} B_2 \,, \tag{II.23}$$

where q_{F1} is the flux of the electric field on a spatial (p-1)-submanifold of Σ_{p+1} and $*$ is the worldvolume Hodge dual.

Indeed, saying that the F_2 is a source for such closed string fields requires a point of view in which the two pieces of the actions (II.22)-(II.23) are added to the (supergravity) closed strings action. If one takes the opposite point of view, then we can say that the D(p-2)/F1 background supergravity fields act as sources of magnetic/electric components of worldvolume gauge field.

These ideas are key to construct a supertube. Imagine taking a large set of F1's in flat space and trying to form a tube $\mathbb{R} \times S^1$ with them. In the continuous limit in which we think of a constant density of F1's along the cross section S^1, this will look like a tubular D2 brane with dissolved q_{F1} charge. The D2 tension will try to collapse the tube, so we could think of trying to stabilize it by making it rotate. However, momentum tangent to the brane directions is unphysical, so we must abandon the idea. However, it is known that we can link any number of D0-branes in an F1 at the cost of breaking a half of the original 1/2 supersymmetries preserved by the string. From the point of view of the D2, the F1's will be described by an electric F_2 and the D0's by a magnetic one. These fields generate a Poynting vector which, after a careful choice of the D0/F1 charge densities, emulates the effect of the necessary angular momentum that prevents the D2 from collapse. Indeed, it was realized in [47] that one can achieve any arbitrary cross section $S^1 \to \mathcal{C}$ and still have a stable supersymmetric object. It is just a matter of choosing the right local charge densities that generate the appropriate Poynting vector; the latter provides a centrifugal force which compensates the effect of the tension *at every point of* $\mathcal{C}$.

Having heuristically explained how to stabilize supersymmetrically a supertube in flat space, we can think of whether there is any chance to do the same in a curved background. After all, the effect of the background curvature is to modify the precise value of the force felt by each point of $\mathcal{C}$ by adding a gravitational effect. Qualitatively it does not look too different from the stabilization of an arbitrary curve in flat space. We will confirm that this is indeed possible. What it looks much more difficult is to be able to do it preserving any supersymmetry at all. Two conditions must be satisfied:

1. the background itself must leave some supersymmetries unbroken,
2. the supercharges preserved by the supertube must be compatible with those preserved by the background.

The first step is then to choose a supersymmetric background of type IIA supergravity. We will only consider cases in which the backgrounds are purely gravitational, so that all fluxes are turned off.[2] Furthermore, we will only consider backgrounds of the form $\mathbb{R}^{1,1} \times \mathcal{M}_8$, with $\mathcal{M}_8$ a curved manifold. This is because we want to put the longitudinal (and the timelike) directions of the tube along the $\mathbb{R}^{1,1}$ factor. The cross section $\mathcal{C}$, however, will completely lie inside $\mathcal{M}_8$ as shown in the figure.

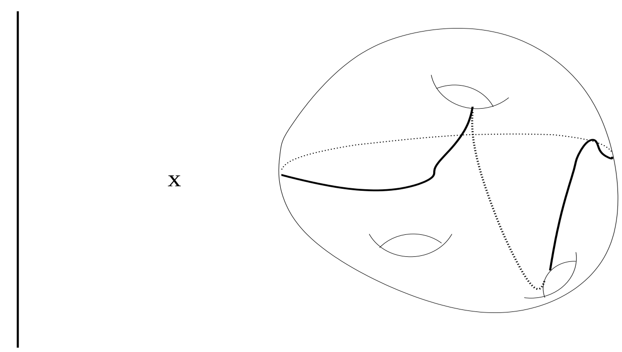


Fig. II.2: The embedding of a supertube in $\mathbb{R} \times \mathcal{M}_8$. The cross section is an arbitrary curve in $\mathcal{M}_8$.

In section IV.5.1 we will discuss in detail the classification of supersymmetric backgrounds of this type. Let us just cite the results here: it turns out that $\mathcal{M}_8$ must be one of the usual manifolds with reduced holonomy [54],

$\mathcal{M}_8$	Fraction of supersymmetry preserved
$\mathbb{R}^4 \times CY_2$	1/2
$CY_2 \times CY_2$	1/4
$\mathbb{R}^2 \times CY_3$	1/4
CY_4	1/8
$\mathbb{R} \times G_2$	1/8
Spin(7)	1/8
Sp(2)	3/8

[2] Note that this excludes the obvious possibility of putting a supertube in the background created by a large number of supertubes, which has been used recently in the study of closed timelike curves in string theory.

where we have indicated the fraction of supersymmetry preserved by $\mathcal{M}_8$; the maximum is 16, in which case $\mathcal{M}_8 = \mathbb{R}^8$.

II.3.3 Plan and summary of the results

We will extend the analysis of [35, 47] and show that it is possible to supersymmetrically embed the supertube in these backgrounds in such a way that its time and longitudinal directions fill the $\mathbb{R}^{1,1}$ factor, while its compact direction can describe an arbitrary curve $\mathcal{C}$ in $\mathcal{M}_8$. The problem will be analyzed in two different descriptions.

- In the first one, we will perform a worldvolume approach by considering a D2 probe in these backgrounds with the mentioned embedding and with an electromagnetic worldvolume gauge field corresponding to the threshold bound state of D0/F1. With the knowledge of some general properties of the Killing spinors of the $\mathcal{M}_8$ manifolds, it will be shown, using its κ-symmetry, that the probe bosonic effective action is supersymmetric. As in flat space supertubes, the only charges and projections involved correspond to the D0-branes and the fundamental strings, while the D2 ones do not appear anywhere. This is why, in all cases, the preserved amount of supersymmetry will be 1/4 of the fraction already preserved by the choice of background.

  Note that, in particular, this allows for configurations preserving a single supercharge, as is shown in one of the examples that we present.[3] In the other example, we exploit the fact that the curve $\mathcal{C}$ can now wind around the non-trivial cycles that the $\mathcal{M}_8$ manifolds have, and construct a supertube with cylindrical shape $\mathbb{R} \times S^1$, with the S^1 wrapping one of the non-trivial S^2 cycles of an ALE space. In the absence of D0 and F1 charges, q_0 and q_s respectively, the S^1 is a collapsed point in one of the poles of the S^2. As $|q_0 q_s|$ is increased, the S^1 slides down towards the equator. Unlike in flat space, here $|q_0 q_s|$ is bounded from above and it acquires its maximum value precisely when the S^1 is a maximal circle inside the S^2.

- The second approach will be a spacetime description, where the backreaction of the system will be taken into account, and we will be able to describe the configuration by means of a supersymmetric solution

[3] This is not in contradiction with the fact that the minimal spinors in 2+1 dimensions have 2 independent components since, because of the non-vanishing electromagnetic field, the theory on the worldvolume of the D2 is not Lorentz invariant.

of type IIA supergravity, the low-energy effective theory of the closed string sector. Such solutions can be obtained from the original ones, found in [55], by simply replacing the 8-dimensional Euclidean space that appears in the metric by $\mathcal{M}_8$. We will show that this change is consistent with the supergravity equations of motion as long as the various functions and one-forms that were harmonic in $\mathbb{E}^8$ are now harmonic in $\mathcal{M}_8$. It will also be shown that the supergravity solution preserves the same amount of supersymmetry that was found by the probe analysis.

The organization of the analysis is as follows: in section II.3.4 we analyze the system where the D2-supertube probes the $\mathbb{R}^{1,1} \times \mathcal{M}_8$, and prove that the effective worldvolume action for the D2 is supersymmetric using the κ-symmetry. In section II.3.5 we perform the Hamiltonian analysis of the system. We show that the supersymmetric embeddings minimize the energy for given D0 and F1 charges, showing that the tension and the gravitational force induced by the background are locally compensated by the Poynting vector. In section II.3.6 we give to examples in order to clarify and illustrate these constructions. Section II.3.7 is devoted to the supergravity analysis of the generalised supertubes. We prove there the supersymmetry from a spacetime point of view. Conclusions are given in section II.3.8.

II.3.4 Probe worldvolume analysis

In this section we will prove that the curved direction of a supertube can live in any of the usual manifolds with reduced holonomy, while still preserving some amount of supersymmetry. The analysis will be based on the κ-symmetry properties of the bosonic worldvolume action, and its relation with the supersymmetry transformation of the background fields.

II.3.4.1 The setup

Let us write the target space metric on $\mathbb{R}^{1,1} \times \mathcal{M}_8$ as

$$\mathrm{d}s^2_{IIA} = -(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2 + e^{\underline{i}} e^{\underline{j}} \delta_{\underline{ij}} \,, \qquad e^{\underline{i}} = \mathrm{d}y^j e_j{}^{\underline{i}} \,, \qquad i,j = 2,3,...,9 \,, \tag{II.24}$$

where $e^{\underline{i}}$ is the vielbein of a Ricci-flat metric on $\mathcal{M}_8$. Underlined indices refer to tangent space objects. We will embed the supertube in such a way that its time and longitudinal directions live in $\mathbb{R}^{1,1}$ while its curved direction describes an arbitrary curve $\mathcal{C}$ in $\mathcal{M}_8$. By naming the D2 worldvolume

coordinates $\{\sigma^0, \sigma^1, \sigma^2\}$, such an embedding is determined by

$$x^0 = \sigma^0, \qquad x^1 = \sigma^1, \qquad y^i = y^i(\sigma^2)\,, \tag{II.25}$$

where y^i are arbitrary functions of σ^2. Let us remark that, in general, the curve $\mathcal{C}$ will be contractible in $\mathcal{M}_8$. As a consequence, due to its tension, the compact direction of the D2 will naturally tend to collapse to a point.

Following [35], we will stabilize the D2 by turning on an electromagnetic flux in its worldvolume

$$F_2 = E\,\mathrm{d}\sigma^0 \wedge \mathrm{d}\sigma^1 + B\,\mathrm{d}\sigma^1 \wedge \mathrm{d}\sigma^2\,, \tag{II.26}$$

which will provide the necessary centrifugal force to compensate the D2 tension and the gravitational effect due to the curvature of $\mathcal{M}_8$. We will restrict to static configurations.

The effective action of the D2 is the DBI action (the Wess-Zumino term vanishes in our purely geometrical backgrounds),

$$S = \int_{\mathbb{R}^{1,1}\times C} \mathrm{d}\sigma^0\mathrm{d}\sigma^1\mathrm{d}\sigma^2\mathcal{L}_{DBI}\,, \qquad \mathcal{L}_{DBI} = -\Delta \equiv -\sqrt{-\det[g+F]}\,, \tag{II.27}$$

where g is the induced metric determined by the embedding $x^M(\sigma^\mu)$, and $F_{\mu\nu}$ is the electromagnetic field strength. M denotes the spacetime components $0, 1, \ldots, 9$, and μ labels the worldvolume coordinates $\mu = 0, 1, 2$. The κ-symmetry imposes restrictions on the background supersymmetry transformation when only worldvolume bosonic configurations are considered. Basically we get $\Gamma_\kappa \epsilon = \epsilon$ (see e.g. [56]), where ϵ is the background Killing spinor and Γ_κ (see e.g. [57]) is a matrix that squares to 1:

$$\mathrm{d}^3\sigma\;\Gamma_\kappa = \Delta^{-1}\left[\gamma_3 + \gamma_1\Gamma_* \wedge F_2\right]. \tag{II.28}$$

Here Γ_* is the chirality matrix in ten dimensions (in our conventions it squares to one), and the other definitions are

$$\begin{aligned} \gamma_3 &= \mathrm{d}\sigma^0 \wedge \mathrm{d}\sigma^1 \wedge \mathrm{d}\sigma^2\, \partial_0 x^M \partial_1 x^N \partial_2 x^P e_M{}^{\underline{M}} e_N{}^{\underline{N}} e_P{}^{\underline{P}} \Gamma_{\underline{MNP}}\,, \\ \gamma_1 &= \mathrm{d}\sigma^\mu \partial_\mu x^M e_M{}^{\underline{M}} \Gamma_{\underline{M}}\,. \end{aligned} \tag{II.29}$$

where $e_M{}^{\underline{M}}$ are the vielbeins of the target space and $\Gamma_{\underline{M}}$ are the flat gamma matrices. We are using Greek letters for worldvolume indices and Latin characters for the target space.

We are now ready to see under which circumstances can the configuration (II.25), (II.26) be supersymmetric. This is determined by the condition for κ-symmetry, which becomes

$$[\Gamma_{\underline{01}}\gamma_2 + E\gamma_2\Gamma_* + B\Gamma_{\underline{0}}\Gamma_* - \Delta]\epsilon = 0\,, \tag{II.30}$$

where

$$\Delta^2 = B^2 + y'^{\underline{i}} y'^{\underline{i}} (1 - E^2)\,, \qquad y'^{\underline{i}} = y'^{i} e_i{}^{\underline{i}}\,, \qquad \gamma_2 = y'^{\underline{i}} \Gamma_{\underline{i}}\,, \qquad y'^{i} := \partial_2 y^i\,. \tag{II.31}$$

The solutions of (II.30) for ϵ are the Killing spinors of the background, determining the remaining supersymmetry.

II.3.4.2 Proof of worldvolume supersymmetry

In this section we shall prove that the previous configurations always preserve 1/4 of the remaining background supersymmetries preserved by the choice of $\mathcal{M}_8$. We will show that the usual supertube projections are necessary and sufficient in all cases except when we do not require that the curve $\mathcal{C}$ is arbitrary and it lies completely within the flat directions that $\mathcal{M}_8$ may have. Therefore we first discuss the arbitrary case, and after that, we deal with the special situation.

Arbitrary Curve: If we demand that the configuration is supersymmetric for any arbitrary curve in $\mathcal{M}_8$, then all the terms in (II.30) that contain the derivatives $y'^{i}(\sigma^2)$ must vanish independently of those that do not contain them. The vanishing of the first ones (those containing γ_2) give

$$\Gamma_{\underline{01}} \Gamma_* \epsilon = -E\epsilon \qquad \Longrightarrow \qquad E^2 = 1\,, \qquad \text{and} \qquad \Gamma_{\underline{01}} \Gamma_* \epsilon = -\,\mathrm{sign}(E)\epsilon\,, \tag{II.32}$$

which signals the presence of fundamental strings in the longitudinal direction of the tube. Now, when $E^2 = 1$, then $\Delta = |B|$, and the vanishing of the terms independent of $y'^{i}(\sigma^2)$ in (II.30) give

$$\Gamma_{\underline{0}} \Gamma_* \epsilon = \mathrm{sign}(B)\epsilon\,, \tag{II.33}$$

which signals the presence of D0 branes dissolved in the worldvolume of the supertube. Since both projections, (II.32) and (II.33), commute, the configuration will preserve 1/4 of the background supersymmetries *as long as they also commute with all the projections imposed by the background itself.*

It is easy to prove that this will always be the case. Since the target space is of the form $\mathbb{R}^{1,1} \times \mathcal{M}_8$, the only nontrivial conditions that its Killing spinors have to fulfil are

$$\nabla_i \epsilon = \left(\partial_i + \frac{1}{4} w_i{}^{\underline{jk}} \Gamma_{\underline{jk}} \right) \epsilon = 0\,, \tag{II.34}$$

with all indices only on $\mathcal{M}_8$ (which in our ordering, means $2 \leq i \leq 9$). If one prefers, the integrability condition can be written as

$$[\nabla_i, \nabla_j]\epsilon = \frac{1}{4} R_{ij}{}^{\underline{kl}} \Gamma_{\underline{kl}} \epsilon = 0\,. \tag{II.35}$$

In either form, all the conditions on the background spinors involve only a sum of terms with two (or none) gamma matrices of $\mathcal{M}_8$. It is then clear that such projections will always commute with the F1 and the D0 ones, since they do not involve any gamma matrix of $\mathcal{M}_8$.

To complete the proof, one must take into account further possible problems that could be caused by the fact that the projections considered so far are applied to background spinors which are not necessarily constant. To see that this does not change the results, note that (II.34) implies that all the dependence of ϵ on the $\mathcal{M}_8$ coordinates y^i must be of the form

$$\epsilon = M(y)\epsilon_0\,, \tag{II.36}$$

with ϵ_0 a constant spinor, and $M(y^i)$ a matrix that involves only products of even number of gamma matrices on $\mathcal{M}_8$ (it may well happen that $M(y) = \mathbb{1}$). Now, any projection on ϵ can be translated to a projection on ϵ_0 since

$$\begin{aligned} &P\epsilon = \epsilon\,, \qquad \text{with} \qquad P^2 = \mathbb{1}\,, \qquad \mathrm{Tr}\, P = 0\,, \qquad \Longrightarrow \\ &\tilde{P}\epsilon_0 = \epsilon_0\,, \qquad \text{with} \qquad \tilde{P} \equiv M^{-1}(y) P M(y)\,, \quad \tilde{P}^2 = \mathbb{1}\,, \quad \mathrm{Tr}\, \tilde{P} = 0\,. \end{aligned} \tag{II.37}$$

The only subtle point here is that, if some of the ϵ_0 have to survive, the product of $M^{-1}(y)PM(y)$ must be a constant matrix[4]. But this is always the case for all the projections related to the presence of $\mathcal{M}_8$, since we know that such spaces preserve some Killing spinors. Finally, it is also the case for the F1 and D0 projections, since they commute with any even number of gamma matrices on $\mathcal{M}_8$.

The conclusion is that, for an arbitrary curve in $\mathcal{M}_8$ to preserve supersymmetry, it is necessary and sufficient to impose the F1 and D0 projections. In all cases, it will preserve 1/4 of the background supersymmetry. We will illustrate this with particular examples in section II.3.6.

Non-Arbitrary Curve: If we now give up the restriction that the curve must be arbitrary, we can still show that the F1 and D0 projection

[4] Note that it is not necessary that P commutes with $M(y)$.

are necessary and sufficient, except for those cases in which the curve lies entirely in the flat directions that $\mathcal{M}_8$ may have. Of course, the former discussion shows that such projections are always sufficient, so we will now study in which cases they are necessary as well.

In order to proceed, we need to prove an intermediate result.

Lemma: If the velocity of the curve does not point in a flat direction of $\mathcal{M}_8$, then the background spinor always satisfies at least one projection like

$$P\epsilon = Q\epsilon\,, \qquad \textit{such that} \qquad [P,\gamma_2] = 0\,, \qquad \{Q,\gamma_2\} = 0\,, \tag{II.38}$$

with P and Q a non-vanishing sum of terms involving only an even number of gamma matrices, and Q invertible.

To prove this, we move to a point of the curve that lies in a curved direction of $\mathcal{M}_8$, i.e. a point where not all components of $R_{ij}{}^{\underline{kl}}$ are zero. We perform a rotation in the tangent space such that the velocity of the curve points only in one of the curved directions, *e.g.*

$$y'^{\underline{9}} \neq 0\,, \qquad y'^{\underline{a}} = 0\,, \qquad a = 2,...,8\,, \qquad R_{ij}{}^{a9} \neq 0\,, \tag{II.39}$$

for at least one choice of i, j and a, and where we use the definitions of (II.31). With this choice, γ_2 becomes simply $\gamma_2 = y'^{\underline{9}}\Gamma_{\underline{9}}$. Therefore, at least one of the equations in (II.35) can be split in

$$\left(R_{ij}{}^{\underline{ab}}\Gamma_{\underline{ab}} + R_{ij}{}^{\underline{a9}}\Gamma_{\underline{a9}}\right)\epsilon = 0\,, \tag{II.40}$$

with the definitions

$$P = R_{ij}{}^{\underline{ab}}\Gamma_{\underline{ab}}\,, \qquad Q = -R_{ij}{}^{\underline{a9}}\Gamma_{\underline{a9}}\,. \tag{II.41}$$

The assumption (II.39) implies that Q is nonzero and invertible, as the square of Q is a negative definite multiple of the unit matrix. This implies that also P is non-zero since, otherwise, ϵ would have to be zero and this is against the fact that all the listed $\mathcal{M}_8$ manifolds admit covariantly constant spinors. It is now immediate to check that γ_2 commutes with P while it anticommutes with Q, which completes the proof. ■

We can now apply this lemma and rewrite one of the conditions in (II.35) as an equation of the kind (II.38). We then multiply the κ-symmetry condition (II.30) by $P-Q$. Clearly only the first two terms survive, and we can write

$$0 = [\Gamma_{\underline{01}} - E\Gamma_*]\,(P-Q)\gamma_2\epsilon = -2\,[\Gamma_{\underline{01}} - E\Gamma_*]\,\gamma_2 Q\epsilon = -2\gamma_2 Q\,[\Gamma_{\underline{01}} + E\Gamma_*]\,\epsilon\,. \tag{II.42}$$

Since $(\gamma_2)^2 = y'_i y'^i$ cannot be zero if the curve is not degenerate, we just have to multiply with $Q^{-1}\gamma_2$ to find again (II.32). Plugging this back into (II.30) gives the remaining D0 condition (II.33).

Summarizing, the usual supertube conditions are always necessary and sufficient except for those cases where the curve is not required to be arbitrary and lives entirely in flat space; then, they are just sufficient. For example, one could choose $\mathcal{C}$ to be a straight line in one of the $\mathbb{R}$ factors that some of the $\mathcal{M}_8$have, and take a constant B, which would correspond to a planar D2-brane preserving 1/2 of the background supersymmetry.

II.3.5 Hamiltonian analysis

We showed that in order for the supertube configurations (II.25), (II.26) to be supersymmetric we needed $E^2 = 1$, but we found no restriction on the magnetic field $B(\sigma^1, \sigma^2)$. We shall now check that some conditions must hold in order to solve the equations of motion of the Maxwell fields. We will go through the Hamiltonian analysis which will enable us to show that these supertubes saturate a BPS bound which, in turn, implies the second-order Lagrange equations on the submanifold determined by the constraints. We will restrict to time-independent configurations, which we have checked to be compatible with the full equations of motion. The Lagrangian is then given by (II.27)

$$\mathcal{L} = -\Delta = -\sqrt{B^2 + R^2(1 - E^2)}\,, \tag{II.43}$$

where we have defined $R^2 = y'^{\underline{i}} y'_{\underline{i}}$, and $R > 0$. To obtain the Hamiltonian we first need the displacement field,

$$\Pi = \frac{\partial \mathcal{L}}{\partial E} = \frac{E R^2}{\sqrt{B^2 + (1 - E^2) R^2}}\,, \tag{II.44}$$

which can be inverted to give

$$E = \frac{\Pi}{R}\sqrt{\frac{B^2 + R^2}{R^2 + \Pi^2}}\,, \qquad \Delta = R\sqrt{\frac{B^2 + R^2}{R^2 + \Pi^2}}\,. \tag{II.45}$$

The Lagrange equations for A_0 and A_2 give two constraints

$$\partial_1 \Pi = 0\,, \qquad \partial_1\left(\frac{B}{R}\sqrt{\frac{R^2 + \Pi^2}{B^2 + R^2}}\right) = 0\,, \tag{II.46}$$

the first one being the usual Gauss law. Together, they imply that $\partial_1 B = 0$, i.e., the magnetic field can only depend on σ^2. Finally, the equations for A_1 and $y^{\underline{i}}$ give, respectively,

$$\partial_2 \left(\frac{B}{R} \sqrt{\frac{R^2 + \Pi^2}{B^2 + R^2}} \right) = 0 \,, \qquad \partial_2 \left[2 y'^{\underline{i}} \frac{R^4 - \Pi^2 B^2}{R^2 \sqrt{(R^2 + \Pi^2)(R^2 + B^2)}} \right] = 0 \,. \tag{II.47}$$

The Hamiltonian density is given by

$$\mathcal{H} = E\Pi - \mathcal{L} = \frac{1}{R} \sqrt{(R^2 + \Pi^2)(B^2 + R^2)} \,. \tag{II.48}$$

In order to obtain a BPS bound [58], we rewrite the square of the Hamiltonian density as

$$\mathcal{H}^2 = (\Pi \pm B)^2 + \left(\frac{\Pi B}{R} \mp R \right)^2 \,, \tag{II.49}$$

from which we obtain the local inequality

$$\mathcal{H} \geq |\Pi \pm B| \,, \tag{II.50}$$

which can be saturated only if

$$R^2 = y'^{\underline{i}} y'_{\underline{i}} = \pm \Pi B \qquad \Leftrightarrow \qquad E^2 = 1 \,. \tag{II.51}$$

It can be checked that the configurations saturating this bound satisfy the remaining equations of motion (II.47).

Note that the Poynting vector generated by the electromagnetic field is always tangent to the curve $\mathcal{C}$ and its modulus is precisely $|\Pi B|$. We can then use exactly the same arguments as in [47]. Equation (II.51) tells us that, once we set $E^2 = 1$, and regardless of the value of $B(\sigma^2)$, the Poynting vector is automatically adjusted to provide the required centripetal force that compensates both the tension and the gravitational effect due to the background curvature at every point of $\mathcal{C}$. The only difference with respect to the original supertubes in flat space is that the curvature of the background is taken into account in (II.51), through the explicit dependence of R^2 on the metric of $\mathcal{M}_8$.

Finally, the integrated version of the BPS bound (II.50) is

$$\tau \geq |q_0 \pm q_s| \,, \tag{II.52}$$

with

$$\tau \equiv \int_{\mathcal{C}} \mathrm{d}\sigma^2 \, \mathcal{H} \,, \qquad q_0 \equiv \int_{\mathcal{C}} \mathrm{d}\sigma^2 \, B \,, \qquad q_s \equiv \int_{\mathcal{C}} \mathrm{d}\sigma^2 \, \Pi \,. \tag{II.53}$$

and the normalization $0 \leq \sigma^2 < 1$. Similarly, the integrated bound is saturated when

$$L(\mathcal{C}) = \int_{\mathcal{C}} \mathrm{d}\sigma^2 \sqrt{g_{22}} = \int_{\mathcal{C}} \mathrm{d}\sigma^2 \sqrt{y'^{\underline{i}} y'_{\underline{i}}} = \int_{\mathcal{C}} \mathrm{d}\sigma^2 \sqrt{|\Pi B|} = \sqrt{|q_s\, q_0|}\,, \quad \text{(II.54)}$$

where $L(\mathcal{C})$ is precisely the proper length of the curve $\mathcal{C}$, and the last equality is only valid when both Π and B are constant, as will be the case in our examples.

II.3.6 Examples

After having discussed the general construction of supertubes in reduced holonomy manifolds, we shall now present two examples in order to illustrate some of their physical features.

II.3.6.1 Supertubes in ALE spaces: 4 supercharges

Let us choose $\mathcal{M}_8 = \mathbb{R}^4 \times CY_2$, i.e. the full model being $\mathbb{R}^{1,5} \times CY_2$. We take the CY_2 to be an ALE space provided with a multi-Eguchi–Hanson metric [59]

$$\begin{aligned} \mathrm{d}s^2_{(4)} &= V^{-1}(\vec{y})\mathrm{d}\vec{y}\cdot\mathrm{d}\vec{y} + V(\vec{y})\left(\mathrm{d}\psi + \vec{A}\cdot\mathrm{d}\vec{y}\right)^2\,, \\ V^{-1}(\vec{y}) &= \sum_{r=1}^{N} \frac{Q}{|\vec{y}-\vec{y}_r|}\,, \qquad\qquad \vec{\nabla}\times\vec{A} = \vec{\nabla}V^{-1}(\vec{y})\,, \quad \text{(II.55)} \end{aligned}$$

with $\vec{y} \in \mathbb{R}^3$. These metrics describe a U(1) fibration over $\mathbb{R}^3$, the circles being parametrized by $\psi \in [0,1]$. They present N removable bolt singularities at the points $\vec{y}_r$, where the U(1) fibres contract to a point. Therefore, a segment connecting any two such points, together with the fibre, form (topologically) an S^2. For simplicity, we will just consider the two-monopoles case which, without loss of generality, can be placed at $\vec{y} = \vec{0}$ and $\vec{y} = (0,0,b)$. Therefore, the complete IIA background is

$$\mathrm{d}s^2_{IIA} = -(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2 + ... + (\mathrm{d}x^5)^2 + \mathrm{d}s^2_{(4)}\,, \quad \text{(II.56)}$$

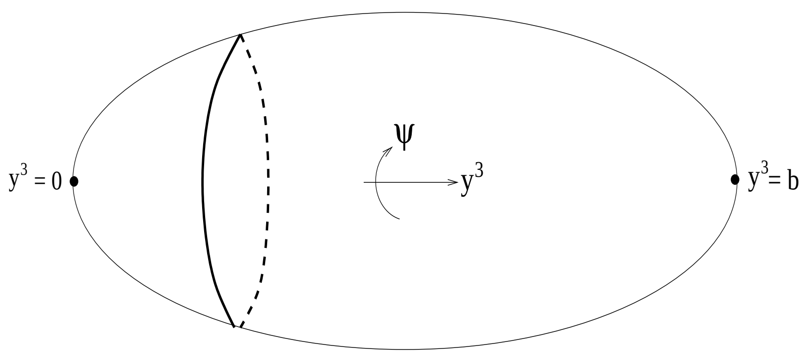
with

$$V^{-1}(\vec{y}) = \frac{Q}{|\vec{y}|} + \frac{Q}{|\vec{y}-(0,0,b)|}\,. \quad \text{(II.57)}$$

Let us embed the D2 supertube in a way such that its longitudinal direction lies in $\mathbb{R}^5$ while its compact one wraps and S^1 inside the S^2 that connects

the two monopoles. More explicitly,

$$X^0 = \sigma^0\,, \qquad X^1 = \sigma_1\,, \qquad \psi = \sigma^2\,, \qquad y^3 = \text{const.}\,, \qquad y^1 = y^2 = 0\,. \tag{II.58}$$



Since any S^1 is contractible inside an S^2, the curved part would tend collapse to the nearest pole, located at $y^3 = 0$ or $y^3 = b$. As in flat space, we therefore need to turn on a worldvolume flux as in (II.26), with E and B constant for the moment.

According to our general discussion, this configuration should preserve 1/4 of the 16 background supercharges already preserved by the ALE space. In this case, the κ-symmetry equation is simply

$$\Big(\Gamma_{\underline{01\psi}} + E\Gamma_{\underline{\psi}}\Gamma_* + B\Gamma_{\underline{0}}\Gamma_* - \Delta\Big)\,\epsilon = 0\,, \tag{II.59}$$

where ϵ are the Killing spinors of the background (II.56). They can easily be computed and shown to be just constant spinors subject to the projection

$$\Gamma_{\underline{y^1y^2y^3\psi}}\,\epsilon = -\epsilon\,. \tag{II.60}$$

Then, the κ-symmetry equation can be solved by requiring (II.32) and (II.33), which involve the usual D0/F1 projections of the supertube. Since they commute with (II.60), the configuration preserves a total of 1/8 of the 32 supercharges.

It is interesting to see what are the consequences of having $E^2 = 1$ for this case. Note that, from our general Hamiltonian analysis, we saw that, for fixed D0 and F1 charges, the energy is minimized for $E^2 = 1$. When applied to the present configuration, (II.54) reads

$$V(y^3) = |q_0 q_s|\,, \tag{II.61}$$

which determines y^3, and therefore selects the position of the S^1 inside the S^2 that is compatible with supersymmetry. Since $V(y^3)$ is invariant under $y^3 \leftrightarrow (b - y^3)$, the solutions always come in mirror pairs with respect to the equator of the S^2. The explicit solutions are indeed

$$y^3_{\pm} = \frac{b}{2}\left(1 \pm \sqrt{1 - \frac{4Q}{b}|q_0 q_s|}\right) . \tag{II.62}$$

Note that a solution exists as long as the product of the charges is bounded from above to

$$|q_0 q_s| \leq \frac{b}{4Q} . \tag{II.63}$$

The point is that this will always happen due to the fact that, contrary to the flat space case, the S^1 cannot grow arbitrarily within the S^2. As a consequence, the angular momentum acquires its maximum value when the S^1 is precisely in the equator. To see it more explicitly, setting $E^2 = 1$ and computing q_0 and q_s for our configuration gives

$$|q_0 q_s| = V(y^3) \leq V(y^3 \to \frac{b}{2}) = \frac{b}{4Q} , \tag{II.64}$$

which guarantees that (II.63) is always satisfied.

Finally, note that we could have perfectly chosen, for instance, a more sophisticated embedding in which y^3 was not constant. This would be the analogue of taking a non-constant radius in the original flat space supertube. Again, by the general analysis of the previous sections, this would just require the Poynting vector to vary accordingly in order to locally compensate for both the tension and the gravitational effect due to the background everywhere, and no further supersymmetry would be broken.

II.3.6.2 Supertubes in CY_4 spaces: 1 supercharge

The purpose of the next example is to show how one can reach a configuration with one single surviving supercharge in a concrete example. One could take any of the 1/8-preserving backgrounds of the $\mathcal{M}_8$ Table. Many metrics for these spaces have been recently found in the context of supergravity duals of non-maximally supersymmetric field theories. Let us take the CY_4 that was found in [39, 60] since the Killing spinors have been already calculated explicitly [42][5]. This space is a C^2 bundle over $S^2 \times S^2$,

[5] The construction of this space and its Killing spinors is included in this thesis, sections IV.8 and VI.4.

and the metric is

$$\begin{aligned} ds^2_{(CY_4)} &= A(r)\left[\mathrm{d}\theta_1^2+\sin^2\theta_1\mathrm{d}\phi_1^2+\mathrm{d}\theta_2^2+\sin^2\theta_2\mathrm{d}\phi_2^2\right] \\ &+U^{-1}\mathrm{d}r^2+\frac{r^2}{4}\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right) \\ &+\frac{1}{4}Ur^2\left(\mathrm{d}\psi+\cos\theta\mathrm{d}\phi+\cos\theta_1\mathrm{d}\phi_1+\cos\theta_2\mathrm{d}\phi_2\right)^2 , \end{aligned} \tag{II.65}$$

where

$$A(r)=\frac{3}{2}(r^2+l^2)\,,\qquad U(r)=\frac{3r^4+8l^2r^2+6l^4}{6(r^2+l^2)^2}\,,\qquad C(r)=\frac{1}{4}U\,r^2\,. \tag{II.66}$$

By writing the complete IIA background metric as

$$\mathrm{d}s^2_{IIA}=-(\mathrm{d}x^0)^2+(\mathrm{d}x^1)^2+\mathrm{d}s^2_{(CY_4)}\,, \tag{II.67}$$

and using the obvious vielbeins, with the order

$$\begin{array}{cccccccc} 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \theta_1 & \theta_2 & \phi_2 & \phi_1 & r & \theta & \phi & \psi \end{array} \tag{II.68}$$

the corresponding Killing spinors are

$$\epsilon=\mathrm{e}^{-\frac{1}{2}\psi\Gamma_{\underline{78}}}\epsilon_0\,, \tag{II.69}$$

with ϵ_0 a constant spinor subject to

$$\Gamma_{\underline{25}}\epsilon_0=\Gamma_{\underline{34}}\epsilon_0\,,\qquad \Gamma_{\underline{25}}\epsilon_0=\Gamma_{\underline{78}}\epsilon_0\,,\qquad \Gamma_{\underline{67}}\epsilon_0=\Gamma_{\underline{98}}\epsilon_0\,. \tag{II.70}$$

To analyze κ-symmetry, let us take the compact part of the supertube to lie along, say, the ϕ_1 direction, while setting to constant the rest of the CY_4 coordinates. As in the previous example, this would have the interpretation of an S^1 embedding in one of the two S^2 in the base of the CY_4. Imposing κ-symmetry:

$$\left(\Gamma_{\underline{015}}+E\Gamma_{\underline{5}}\Gamma_*+B\Gamma_{\underline{0}}\Gamma_*-\Delta\right)\epsilon=0\,. \tag{II.71}$$

Now, the first projection of (II.70) happens to anticommute with the γ_2 defined in (II.31)

$$\gamma_2=y'^{i}e_i{}^{\underline{i}}\Gamma_{\underline{i}}\;=\;A^{\frac{1}{2}}(r)\,\sin\theta_1\,\Gamma_{\underline{5}}\,. \tag{II.72}$$

In other words, this just illustrates a particular case of (II.38) for which the direction $\underline{5}$ plays the role of $\underline{9}$, and for which $P=\Gamma_{\underline{34}}$ and $Q=\Gamma_{\underline{25}}$. We can now follow the steps in section II.3.4.2 and multiply (II.71) by $P-Q$. This yields again the usual supertube conditions (II.32) and (II.33).

Since all the gamma matrices appearing in (II.70), (II.32) and (II.33) commute, square to one and are traceless, the configuration preserves only one of the 32 supercharges of the theory. Of course, this is not in contradiction with the fact that the minimal spinors in 2+1 dimensions have 2 components, since the field theory on the worldvolume of the D2 is not Lorentz invariant because of the non-vanishing electromagnetic field.

II.3.7 Supergravity analysis

In this section we construct the supergravity family of solutions that correspond to all the configurations studied above. We start our work with a generalization of the ansatz used in [55, 47] to find the original solutions in flat space. Our analysis is performed in eleven dimensional supergravity, mainly because its field content is much simpler than in IIA supergravity. Once the eleven-dimensional solution is found, we reduce back to ten dimensions, obtaining our generalised supertube configurations.

The first step in finding the solutions is to look for supergravity configurations with the isometries and supersymmetries suggested by the worldvolume analysis of the previous sections. Then, we will turn to the supergravity field equations to find the constraints that the functions of our ansatz have to satisfy in order that our configurations correspond to minima of the eleventh dimensional action. Finally, we choose the correct behavior for these functions so that they correctly describe the supertubes once the reduction to ten dimensions is carried on.

II.3.7.1 Supersymmetry analysis

Our starting point is the supertube ansatz of [55, 47]

$$\begin{aligned} \mathrm{d}s^2_{\mathit{10}} &= -U^{-1}V^{-1/2}\,(\mathrm{d}t-A)^2+U^{-1}V^{1/2}\,\mathrm{d}x^2+V^{1/2}\,\delta_{ij}\mathrm{d}y^i\mathrm{d}y^j\,, \\ B_{\mathit{2}} &= -U^{-1}\,(\mathrm{d}t-A)\wedge\mathrm{d}x+\mathrm{d}t\wedge\mathrm{d}x\,, \\ C_{\mathit{1}} &= -V^{-1}\,(\mathrm{d}t-A)+\mathrm{d}t\,, \\ C_{\mathit{3}} &= -U^{-1}\mathrm{d}t\wedge\mathrm{d}x\wedge A\,, \\ e^{\phi} &= U^{-1/2}V^{3/4}\,, \end{aligned} \tag{II.73}$$

where the Euclidean space ($\mathbb{E}_8$) coordinates are labelled by y^i, with $i,j,\dots=(2,\dots,9)$, $V=1+K$, $A=A_i\,\mathrm{d}y^i$ and B_{2} and C_p are respectively, the Neveu-Schwarz and Ramond-Ramond potentials. V,U,A_i depend only on the $\mathbb{E}_8$ coordinates.

To up-lift this ansatz, we use the normal Kaluza-Klein form of the eleven dimensional metric and three-form,

$$\begin{aligned} \mathrm{d}s^2_{11} &= e^{-2\phi/3}\mathrm{d}s^2_{10} + e^{4\phi/3}(\mathrm{d}z + C_1)^2 \,, \\ N_3 &= C_3 + B_2 \wedge \mathrm{d}z \,, \end{aligned} \tag{II.74}$$

where N_3 is the eleventh dimensional three-form. The convention for curved indices is $M = (\mu; i) = (t, z, x\,; 2, 3, ...9)$ and for flat ones $A = (\alpha; a) = (\underline{t}, \underline{z}, \underline{x}\,; \underline{2}, \underline{3}..., \underline{9})$. The explicit form of the eleven-dimensional metric is given by,

$$\begin{aligned} \mathrm{d}s^2_{11} &= U^{-2/3}\left[-\mathrm{d}t^2 + \mathrm{d}z^2 + K(\mathrm{d}t + \mathrm{d}z)^2 + 2(\mathrm{d}t + \mathrm{d}z)A + \mathrm{d}x^2\right] + U^{1/3}\mathrm{d}s^2_8, \\ F_4 &= \mathrm{d}t \wedge \mathrm{d}(U^{-1}) \wedge \mathrm{d}x \wedge \mathrm{d}z - (\mathrm{d}t + \mathrm{d}z) \wedge \mathrm{d}x \wedge \mathrm{d}(U^{-1}A) \,, \end{aligned} \tag{II.75}$$

where $F_4 = \mathrm{d}N_3$. This background is a solution of the equations of motion in eleven dimensions derived from the action

$$S_{11d} = \int \left[R * 1 \,-\, \frac{1}{2}F_4 \wedge *F_4 \,+\, \frac{1}{3}F_4 \wedge F_4 \wedge N_3\right] , \tag{II.76}$$

when the two functions K and U, as well as the one-form A_1, are harmonic in $\mathbb{E}_8$, i.e.,

$$(\mathrm{d} *_8 \mathrm{d})U = 0\,, \quad (\mathrm{d} *_8 \mathrm{d})K = 0\,, \quad (\mathrm{d} *_8 \mathrm{d})A_1 = 0\,, \tag{II.77}$$

where $*_8$ is the Hodge dual with respect to the Euclidean flat metric on $\mathbb{E}^8$. It describes a background with an M2 brane along the directions $\{t, z, x\}$, together with a wave travelling along z, and angular momentum along $\mathbb{E}^8$ provided by A_1.

Next, we generalize the ansatz above by replacing $\mathbb{E}^8$ by one of the eight dimensional $\mathcal{M}_8$ manifolds of the table, and by allowing K, U and A_1 to have an arbitrary dependence on the $\mathcal{M}_8$ coordinates y^i. We therefore replace the previously flat metric on $\mathbb{E}^8$ by a reduced holonomy metric on $\mathcal{M}_8$, with vielbeins $\tilde{e}^a$. Hence, in (II.75), we replace

$$U^{1/3}\delta_{ij}\mathrm{d}y^i\mathrm{d}y^j \qquad \longrightarrow \qquad U^{1/3}\delta_{ab}\tilde{e}^a\tilde{e}^b \,. \tag{II.78}$$

We use a null base of the cotangent space, defined by

$$\begin{aligned} &e^+ = -U^{-2/3}(\mathrm{d}t + \mathrm{d}z)\,, \quad e^- = \frac{1}{2}(\mathrm{d}t - \mathrm{d}z) - \frac{K}{2}(\mathrm{d}t + \mathrm{d}z) - A\,, \\ &e^x = U^{-1/3}\mathrm{d}x\,, \quad e^a = U^{1/6}\tilde{e}^a\,. \end{aligned} \tag{II.79}$$

This brings the metric and F_4 into the form

$$ds^2_{11} = 2e^+e^- + e^xe^x + \delta_{ab}e^ae^b\,, \quad F_4 = -U^{-1}\,\mathrm{d}U\wedge e^x\wedge e^+\wedge e^- - \mathrm{d}A\wedge e^x\wedge e^+\,. \tag{II.80}$$

As customary, the torsion-less condition can be used to determine the spin connection 1-form ω_{AB}. In our null base, the only non-zero components are

$$\begin{aligned}
\omega_{+-} &= -\frac{U_a}{3U}e^a\,,\\
\omega_{+a} &= \frac{1}{2}U^{1/2}\tilde{K}_ae^+ - \frac{U_a}{3U}e^- - \frac{1}{2}a_{ab}e^b\,,\\
\omega_{-a} &= -\frac{U_a}{3U}e^+\,,\\
\omega_{xa} &= -\frac{U_a}{3U}e^x\,,\\
\omega_{ab} &= \frac{U_b}{6U}e^a - \frac{U^a}{6U}e^b + \tilde{\omega}_{ab} + \frac{1}{2}a_{ab}e^+\,,
\end{aligned} \tag{II.81}$$

were we have defined various tensor quantities through the relations

$$\mathrm{d}U = U_ae^a\,, \quad \mathrm{d}K = \tilde{K}_a\tilde{e}^a\,, \quad \mathrm{d}A = \tfrac{1}{2}a_{ab}e^a\wedge e^b\,, \tag{II.82}$$

and $\tilde{\omega}^{bc}$ are the spin connection one-forms corresponding to $\tilde{e}^a$, i.e. $\mathrm{d}\tilde{e}^a + \tilde{\omega}^a{}_b\tilde{e}^b = 0$.

We now want to see under which circumstances our backgrounds preserve some supersymmetry. Since we are in a bosonic background i.e. all the fermions are set to zero, we just need to ensure that the variation of the gravitino vanishes when evaluated on our configurations. In other words, supersymmetry is preserved if there exist nonzero background spinors ϵ such that[6]

$$\left(\partial_A + \frac{1}{4}\omega_A{}^{BC}\Gamma_{BC} - \frac{1}{288}\Gamma_A{}^{BCDE}F_{BCDE} + \frac{1}{36}F_{ABCD}\Gamma^{BCD}\right)\epsilon = 0\,. \tag{II.83}$$

We will try an ansatz such that the spinor depends only on the coordinates on $\mathcal{M}_8$. It is straightforward to write down the eleven equations (II.83) for each value of $A = \{+,-,x,a\}$. The equation for $A = x$ is

$$\frac{U_a}{6U}\Gamma_a\left(\Gamma_x - \Gamma_{+-}\right)\epsilon - \frac{a_{ab}}{12}\Gamma_{ab}\Gamma_-\epsilon = 0\,. \tag{II.84}$$

Assuming that a_{ab} and α_a are arbitrary and independent we find

$$\Gamma_-\,\epsilon = 0\,, \quad \text{and} \quad \Gamma_x\epsilon = -\epsilon\,. \tag{II.85}$$

[6] For the components of p-forms we use the notations of [61].

Using these projections, it is a straightforward algebraic work to see that the equation for $A = +$ and $A = -$ are automatically satisfied. Finally, the equations for $A = a$ simplify to

$$\nabla_i \epsilon \equiv \left(\partial_i + \frac{1}{4} \tilde{\omega}_i{}^{bc} \Gamma_{bc} \right) \epsilon = 0 \,. \qquad \text{(II.86)}$$

By the same arguments as in the previous sections, the projections (II.85) preserve 1/4 of the 32 real supercharges. On the other hand, (II.86) is just the statement that $\mathcal{M}_8$ must admit covariantly constant spinors. Depending on the choice of $\mathcal{M}_8$, the whole 11d background will preserve the expected total number of supersymmetries that we indicated in the table on page 28.

To reduce back to IIA supergravity, we first go to another flat basis

$$e^+ = -U^{-1/3} V^{-1/2} \left(e^0 + e^z \right) \,, \qquad e^- = \tfrac{1}{2} U^{1/3} V^{1/2} \left(e^0 - e^z \right) \,, \qquad \text{(II.87)}$$

which implies that

$$\Gamma_- = U^{-1/3} V^{-1/2} \left(\Gamma_0 - \Gamma_z \right) . \qquad \text{(II.88)}$$

We reduce along z, i.e. replace Γ_z by Γ_*. The projections (II.85) become the usual D0/F1 projections, with the fundamental strings along the x-axis.

$$\Gamma_0 \Gamma_* \epsilon = -\epsilon \,, \quad \text{and} \quad \epsilon = -\Gamma_x \epsilon = \Gamma_{x0} \Gamma_* \epsilon \,. \qquad \text{(II.89)}$$

II.3.7.2 Equations of motion

Now that we have proved that the correct supersymmetry is preserved (matching the worldvolume analysis), we proceed to determine the equations that U, K and A_1 have to satisfy in order that our configurations solve the field equations of eleven-dimensional supergravity. Instead of checking each of the equations of motion, we use the analysis of [62] that is based on the integrability condition derived from the supersymmetry variation of the gravitino (II.83). The result of this analysis is that when at least one supersymmetry is preserved, and the Killing vector $\mathcal{K}_\mu \equiv \bar{\epsilon} \Gamma_\mu \epsilon$ is null, all of the second order equations of motion are automatically satisfied, except for

1. The equation of motion for F_4,
2. The Einstein equation $E_{++} = T_{++}$,

where E_{++} and T_{++} are the Einstein and stress-energy tensors along the components $++$ in a base where $\mathcal{K}_\mu = \delta_\mu^+ \mathcal{K}_+$. Let us explain why the above

statement is correct. The integrability conditions give no information about the field equation for the matter content, therefore the equation of motion for F_4 has to be verified by hand. Also, in most cases all of the Einstein equations are automatically implied by the existence of a non-trivial solution of (II.83).

With (II.85) and in the base where the metric takes the form (II.80), and thus $\Gamma_+\Gamma_- + \Gamma_-\Gamma_+ = 2$, we have

$$\mathcal{K}_\mu = \bar{\epsilon}\Gamma_\mu\epsilon = \tfrac{1}{2}\bar{\epsilon}\Gamma_\mu\Gamma_-\Gamma_+\epsilon\,. \tag{II.90}$$

This vanishes for all μ except $\mu = +$, implying that our configuration falls into the classification of those backgrounds that admit a null Killing spinor and as a consequence the associated Einstein equations escape the analysis. We thus have to check the two items mentioned above.

Let us start with the equation for F_4, which is

$$\mathrm{d} * F_4 + F_4 \wedge F_4 = 0\,. \tag{II.91}$$

Using the fact that the Hodge dual of a p-form with respect to e^a is related to the one with respect to $\tilde{e}^a$ by

$$*_8 C_p = U^{(4-p)/3}\tilde{*}_8 C_p\,, \tag{II.92}$$

where

$$C_p = \frac{1}{p!}C_{a_1\ldots a_p}\tilde{e}^{a_1}\wedge\ldots\wedge\tilde{e}^{a_p} \;\to \tilde{*}_8 C_p = \frac{1}{p!(8-p)!}C_{a_1\ldots a_p}\varepsilon^{a_1\ldots a_8}\tilde{e}^{a_{p+1}}\wedge\ldots\wedge\tilde{e}^{a_8}\,, \tag{II.93}$$

it is easy to see that (II.91) becomes

$$0 = (\mathrm{d}\tilde{*}_8\mathrm{d})U + (\mathrm{d}t + \mathrm{d}z)\wedge(\mathrm{d}\tilde{*}_8\mathrm{d})A\,. \tag{II.94}$$

This implies that U and A_1 must be harmonic with respect to the metric of $\mathcal{M}_8$, i.e.,

$$(\mathrm{d}\tilde{*}_8\mathrm{d})U = 0\,, \quad (\mathrm{d}\tilde{*}_8\mathrm{d})A_1 = 0\,. \tag{II.95}$$

Finally, using (II.80) and (II.81), one can explicitly compute the $\{++\}$ components of the Einstein and stress-energy tensors, and obtain

$$\begin{aligned} E_{++} &= R_{++} = -\tfrac{1}{2}U^{1/3}(\tilde{*}_8\mathrm{d}\tilde{*}_8\mathrm{d})K + \tfrac{1}{2}*_8(\mathrm{d}A\wedge *_8\mathrm{d}A)\,,\\ T_{++} &= \tfrac{1}{12}F_{+ABC}F_+{}^{ABC} = \tfrac{1}{2}*_8(\mathrm{d}A\wedge *_8\mathrm{d}A)\,. \end{aligned} \tag{II.96}$$

Therefore, the last non-trivial equation of motion tells us that also K must be harmonic on $\mathcal{M}_8$,

$$(\mathrm{d}\tilde{*}_8\mathrm{d})K = 0\,. \tag{II.97}$$

II.3.7.3 Constructing the supertube

In order to construct the supergravity solutions that properly describe supertubes in reduced holonomy manifolds, we reduce our eleven-dimensional background to a ten-dimensional background of type IIA supergravity, using (II.74) again. We obtain (II.73) with the replacement (II.78), and the constraints (II.95) and (II.97). At this point we have to choose U, K and A_1 so that they describe a D2-brane with worldvolume $\mathbb{R}^{1,1} \times \mathcal{C}$, with $\mathcal{C}$ an arbitrary curve in $\mathcal{M}_8$. As it was done in [55, 47], one should couple IIA supergravity to a source with support along $\mathbb{R}^{1,1} \times \mathcal{C}$, and solve the $\mathcal{M}_8$ Laplace equations (II.95) and (II.97) with such a source term in the right hand sides. If this has to correspond to the picture of D0/F1 bound states expanded into a D2 by rotation, the boundary conditions of the Laplace equations must be such that the solution carries the right conserved charges. In the appropriate units,

$$q_0 = \int_{\partial\mathcal{M}_8} \tilde{*}_8 \mathrm{d}C_1 \,, \quad q_s = \int_{\partial\mathcal{M}_8} \tilde{*}_8 \mathrm{d}B_2 \,, \quad A_1 \xrightarrow{\partial\mathcal{M}_8} L_{ij} y^j \mathrm{d}y^i \,. \tag{II.98}$$

Here, as in [55, 47], L_{ij} would have to match with the angular momentum carried by the electromagnetic field that we considered in the worldvolume approach.

The Laplace problem in a general manifold can be very complicated and, in most cases, it cannot be solved in terms of ordinary functions. We will not intend to do so, but rather we will just claim that, once U, K and A_1 have been determined, they can be plugged back into (II.73), with (II.78), and the background will describe the configurations that we have been discussing here. It will have the expected isometries, supersymmetries and conserved charges.

II.3.8 Conclusions

We have shown that the expansion of the D0/F1 system into a D2 can happen supersymmetrically in all the backgrounds of the form $\mathbb{R}^{1,1} \times \mathcal{M}_8$, with $\mathcal{M}_8$ any manifold of the table, both in the worldvolume and in the supergravity setting. We would like to stress here that this is not enough to prove that the system is stable in all regimes, as we have only worked in the two mentioned approximations. Indeed, the cross section of the D2-brane can be chosen such that two pieces of D2 that locally look like a pair

D2/anti-D2 are arbitrarily close to each other.[7] This leads to the suspicion that the system develops instabilities against annihilation that may have escaped to our approximations. This led the authors of [47] to explicitly check that no such stability is actually present in the simplest case of a flat D2/anti-D2 pair in flat space. Although there is no hope to carry a similar analysis in our more sophisticated backgrounds, we believe that the same result applies.

We remark that our research is different from [66], where it was shown that *the supertube itself*, after some T-dualities, can be described by a special Lorentzian-holonomy manifold in eleven dimensions.

[7] See [63, 64, 65] for further elaboration on the understanding of supertubes from various dual configurations.

III. ADS/CFT BEYOND SUPERGRAVITY AND SUPERSYMMETRY

This chapter deals with one of the most remarkable dualities derived from string theory: the AdS/CFT correspondence [67, 3, 68]. First, we describe how it was established from D-brane considerations. We pay special attention to carefully settle the regions where both sides of the duality are technically tractable. We then move to the difficult enterprize of trying to find observables that can be computed on both sides, and then compared. This will lead us to introduce the work of Berenstein, Maldacena and Nastase [4], and the shortcut provided by Gubser, Klebanov and Polyakov [6].

The main part of this chapter is section III.4, which contains an expanded discussion of the results presented in [37] and [38]. These papers are devoted to exploiting the ideas of GKP in order to (try to) test the AdS/CFT correspondence away from supergravity and supersymmetry.

Finally, section III.5 contains unfinished work with D. Mateos and P. K. Townsend about the possibility of adding stable but non-supersymmetric matter to superconformal field theories via the AdS/dCFT correspondence.

III.1 The AdS/CFT correspondence

Having discussed the two dual pictures of D-branes in chapter II, we are ready to introduce the ideas of AdS/CFT correspondence, which essentially consist on taking seriously the equivalence of both descriptions.[1] Our discussion will be now centered in the case of D3-branes. The generalization to other Dp-branes will be given in section IV.1.

- Let us consider the open string description. of a set of N D3 probes in the Minkowsky vacuum. The world consists of closed strings oscillating in the 10d space and open strings oscillating with their endpoints

[1] Again, we refer the reader to [69] for a deep and careful discussion.

stuck to the probes. At low energies only the massless excitations matter, and their dynamics are schematically given by

$$S = S_{open} + S_{closed} + S_{int}\,, \tag{III.1}$$

where S_{int} governs the open-closed interactions and all three actions must be understood in the Wilsonian sense, as having integrated out the massive modes. We already know that

$$\begin{aligned} S_{open} &= S\,[\mathcal{N}=4\ \ \text{SYM}] + \mathcal{O}(l_s^2)\,, &\text{(III.2)}\\ S_{closed} &= S\,[\text{IIB SUGRA}] + \mathcal{O}(l_s^2)\,,. &\text{(III.3)}\end{aligned}$$

The most important point is whether we can show that S_{int} is of order l_s or not. What we do know for sure is that $S_{int} = \mathcal{O}(l_P)$ where l_P is the 10d Planck's length

$$l_P \sim g_s l_s^4\,. \tag{III.4}$$

This is because the open-closed interaction starts receiving contributions at diagrams of order g_s (the factor l_s^4 must be there on dimensional grounds). Now the whole subtle point is to realize that (III.2) is meaningful only if we have kept the YM coupling fixed in the $l_s \to 0$ limit. As for our present case of D3 branes we simply have $g_{YM}^2 \sim g_s$, there is no obstruction to having simultaneously

$$l_s \to 0\,, \qquad g_{YM}^2 = \text{fixed}\,, \qquad l_P \to 0\,. \tag{III.5}$$

We will see in sections IV.2.1 and IV.2.2 the problems that this limit originates for D5 and D6 branes, respectively.

The conclusion is that in the zero slope limit of this system we obtain two completely decoupled theories: an $\mathcal{N}=4$ SYM theory plus free supergravity about the flat vacuum.

- We now try to do the same in the closed string picture of the D-branes. The effective action for its massless modes is now just

$$S = S[\text{IIB SUGRA}^*] + \mathcal{O}(l_s^2)\,, \tag{III.6}$$

where by SUGRA* we mean that one considers supergravity excitations about the D3 vacuum solution, given by setting $p = 3$ in (II.13)-(II.15). It was realized that in the $\alpha' \to 0$ limit this geometry becomes disconnected into two regions:

1. Near-Horizon region where $r/l_s^2 = \text{fixed}$.

2. Asymptotic region where r/l_s^2 = unbounded.

By 'disconnected' we mean that, in the limit, no excitation is able to scape from region 1 to region 2 and no excitation from region 2 is able to scatter with those in region 1. The geometry of both regions can be obtained by neglecting either the first or the second term in the function $H = 1 + \frac{R^4}{r^4}$ of the solution. The result is

1. Near-Horizon metric:

$$ds^2 = \frac{R^2}{r^2} dx_{0,3}^2 + \frac{r^2}{R^2} dr^2 + R^2 d\Omega_5^2 \,, \tag{III.7}$$

   which is the metric of $AdS_5 \times S^5$ with both factors having the same radius R.

2. Asymptotic metric: just 10d Minkowsky.

So we are led again to two decoupled systems, the one being supergravity excitations about flat space and the other *any closed string* excitation in $AdS_5 \times S^5$. The way this decoupling was obtained makes it clear the we must be careful with what we mean by *any* in the expression in italics. We cannot allow for almost infinite energy excitations, as they would have never decoupled from the assymptotics region. One is restricted to consider those that do not change the assymptotics of $AdS_5 \times S^5$ space.[2]

Summarizing, we end up with the two points of view having split into two disconnected pieces. As a common factor for both is just closed string excitations about a flat background, one is led to conjecture that the remaining two pieces describe equivalent physics, *i.e.*

$$\boxed{\begin{array}{ccc} \mathcal{N}=4 \text{ SU(N) SYM in 3+1} & = & \text{IIB string theory on } AdS_5 \times S^5 \\ g_{YM}^2 & = & g_s \\ \lambda \equiv g_{YM}^2 N & = & (R_{AdS} \,/\, l_s)^4 \end{array}}$$

[2] This discussion is relevant, for example, in the so-called $AdS/dCFT$ correspondence. There one introduces an infinite D-brane probe in AdS which intersects the boundary. There are then extra degrees of freedom that do not decouple and the dual field theory contains extra matter fields not present in the $\mathcal{N} = 4$ supermultiplet.

Note that we wrote $SU(N)$ instead of $U(N)$ as would be expected from the worldvolume gauge theory considerations of the previous chapter. This is because a $U(N)$ gauge theory is locally equivalent to a $U(1)$ vector multiplet times an $SU(N)$ theory; the $U(1)$ factor decouples from the rest of degrees of freedom. There is however no single field in the supergravity side that does not couple to gravity, so the $U(1)$ degrees of freedom are not expected to be visible as excitations in the bulk of AdS. They are apparently related to the topological theory of B-fields on AdS [70].

This conjecture realizes in an explicit example the old 't Hooft's idea that the degrees of freedom of non-abelian gauge theories could be better described in the non-perturbative phase in terms of stringy degrees of freedom. He showed that the perturbative expansion of correlation functions of SYM theories admits a classification in terms of two parameters: N and $\lambda = g_{YM}^2 N$.[3] For example, the partition function can be written as

$$\mathcal{Z} = \sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda)\,, \qquad f_g(\lambda) = \sum_{n=0}^{\infty} \lambda^n c_{g,n}\,, \tag{III.8}$$

where g is just the genus of a contributing diagram written in double-line notation. In other words, the power of N in each diagram is only controlled by its topology. This resembles very much the role played by g_s in perturbative string theory.

't Hooft had in mind the chance of simplifying the gauge theory by taking N very large. The form of (III.8) tells us how that, in order to keep interactions, this limit must be taken such that

$$\lambda = g_{YM}^2 N = \text{fixed}\,, \qquad N \to \infty\,, \tag{III.9}$$

a limit in which only planar diagrams contribute.

III.1.1 Pre-BMN ranges of validity and comparability

The purpose of this subsection is to make clear what are the regions of the parameter space in both sides of the duality that are under control, and what should we expect away from these regions.

- $\mathcal{N} = 4$ **SYM side.** Although we may not be too familiar with this fact, the effective coupling of an $SU(N)$ gauge theory is not g_{YM}^2 but

[3] As we said, the $\mathcal{N} = 4$ gauge Lagrangian is fixed given g_{YM} and N, but it is more convenient for what follows to consider λ and N as the independent parameters.

$\lambda = g_{YM}^2 N$ (clearly this distinction does not matter for the Standard Model!). So we only have access to *perturbative computations* in the field theory side if we take $\lambda \ll 1$. To go beyond that we would need non-perturbative computations. We stress however that the theory is thought to be well-defined for all λ, for example by its path integral or lattice definition. Note as well that the theory is greatly simplified if we also take $N \gg 1$ since, by (III.8), only the planar contributions survive. Summarizing:

1. Perturbative SYM $\Longleftrightarrow \lambda \ll 1$.
2. Perturbative and Planar SYM $\Longleftrightarrow \lambda \ll 1,\ N \gg 1$.

- **String theory side.** There are three problems one has to face here. The first one is that we do not even have a non-perturbative definition of type IIB string theory. Even if we believe in S-duality, there is a whole gap between the weakly and strongly coupled extremes which is not under control. So the first requirement is more a conceptual one than a computational one: we need $g_s < 1$; translated to YM variables, $\lambda/N < 1$. The second problem is that because of the non-linearity of gravity, the formation of a black hole is typically an inherent non-perturbative process in string theory. As the typical energies of the closed string excitations in $AdS_5 \times S^5$ are of order $E \sim 1/R$, we need to require $1/R \ll 1/l_p$, *i.e.* $N \gg 1$, in order to prevent black hole formation. The third problem is that even under such circumstances we are still unable to quantize the *free* string theory in $AdS_5 \times S^5$ due, among other things, to the presence of RR-fields. Still, if we are ever successful in quantizing it, we know that its low energy limit will lead to IIB *supergravity*, and we know how to deal with it. This approximation will be valid as long as we do not reach energies close to the massive states that we integrated out. This requires $1/R \ll 1/l_s$ which, translating to field theory parameters, is equivalent to $\lambda \gg 1$. Summarizing:

  1. Perturbative String Theory $\Longleftrightarrow \lambda < N\ ,\ N \gg 1$.
  2. Weakly coupled Supergravity $\Longleftrightarrow 1 \ll \lambda < N$.

The important remark is the absolute incompatibility of point 1 in SYM and point 2 in string theory. This constitutes the main obstacle nowadays to make comparisons on both sides. Note as well that if we were able to quantize string theory on $AdS_5 \times S^5$ we would have a huge overlap between perturbative computations on both sides of the duality. We have illustrated

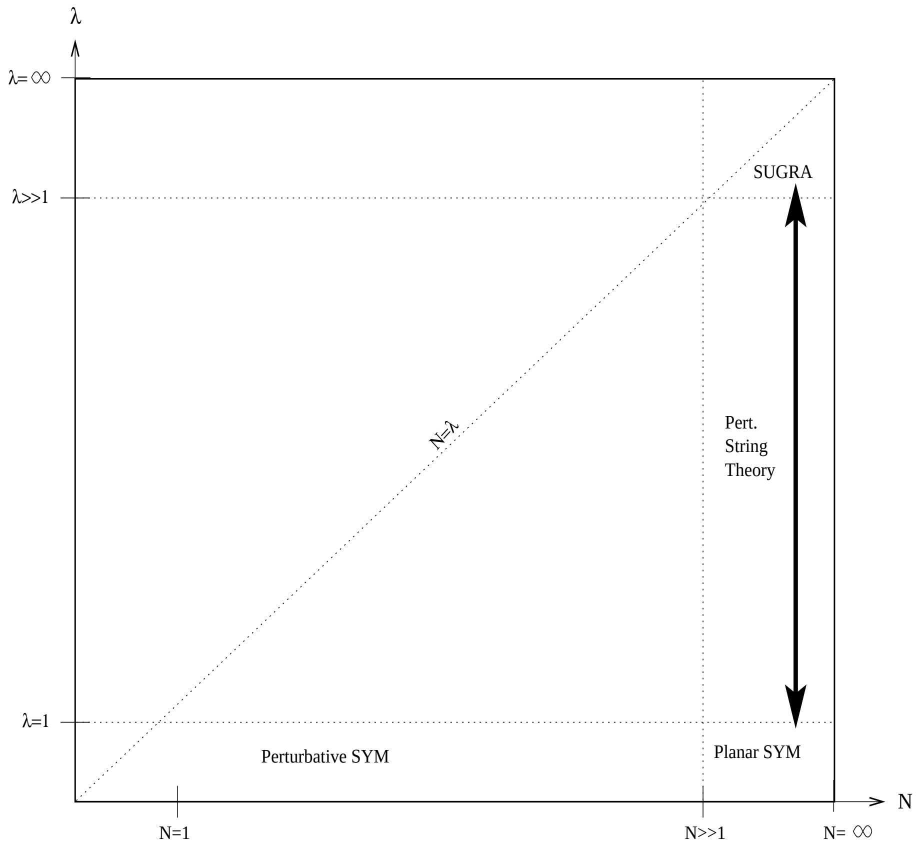


Fig. III.1: The ranges of validity of the AdS/CFT before BMN. A point in this (λ, N)-plane completely determines the SYM Lagrangian. The thick line illustrates that the available *tests* of the correspondence involved extrapolation of SYM observables whose value does not depend on the coupling λ.

the ranges of validity in figure III.1. Having established the computational ranges of validity do not overlap at all, it is natural to ask what can be done with this conjecture. There are essentially two things one may aim to do.

- **Predict.** This is nowadays the less difficult aim. Because of the reasonably well settled dictionary between observables in both sides, one can perform the *same* calculation in the two non-overlapping regions. Both answers remain as a prediction of what the other side should give outside its perturbative domain. This is what we will do in the following chapters, when we will try to show issues like confinement or chiral symmetry breaking for $\mathcal{N} = 1$ and $\mathcal{N} = 2$ gauge theories through their supergravity duals.

- **Compare.** It is clear that this is the most difficult aim as it requires dealing with non-perturbative physics. The picture before the BMN work was that most comparisons had been done under the protection of supersymmetry. For example, the spectrum of superconformal primary operators and their conformal dimensions (these concepts are properly defined in the appendix A) are completely fixed by the superconformal algebra, and one does not need to compute any diagram to determine them. Thus they are independent of the coupling λ and they provide answers that can be compared to weakly coupled supergravity computations. We refer the reader again to [69] for an exhaustive list of successful comparisons in the literature.

  One of the aims of the next sections will be to extend the range of comparability for some observables and to provide new tests of the correspondence. First we will need to recall some properties of the $\mathcal{N}=4$ SYM theory.

III.2 The BMN limit of AdS/CFT

There are some good reviews on the BMN limit of the AdS/CFT correspondence, so we will just describe its basic facts here, specially those that will be relevant in the next sections.

The work of BMN exploited the fact that $AdS_5 \times S^5$ admits a consistent simplification if we just focus on the geometry in the neighborhood of a null geodesic on a great circle of the S^5. An important and non-trivial fact about it is that such a limit produces a configuration that still solves the equations of motion of type IIB supergravity. Furthermore, the number of supersymmetries is still 32. The limit is called a Penrose limit and the resulting background is a maximally supersymmetric pp-wave

$$\begin{aligned} ds^2 &= -4dx^+dx^- - x^i x^i (dx^+)^2 + dx_{1,8}^2\,, \qquad i=1,...,8 &\text{(III.10)}\\ F_5 &= 4\mu\, dx^+ \wedge \left[dx^1\wedge ... \wedge dx^4 + dx^5 \wedge ... \wedge dx^8\right]\,. &\text{(III.11)}\end{aligned}$$

Together with flat space and $AdS_5 \times S^5$, this exhausts the list of maximally supersymmetric IIB backgrounds (see [5] for a proof of this statement). The last important point, and the one that made this line of work so powerful, is that the string σ-model in this background is quantizable despite the presence of nonzero RR fluxes! This opened a new path parallel to the one followed when the string was quantized in flat space: one can obtain the free spectrum, construct the vertex operators and analyze interactions.

The quantizability of the σ-model by itself would not be so remarkable if it was not because the pp-wave background is obtained as a limit of $AdS_5 \times S^5$. After all, there are some few other quantizable backgrounds in the market which have received no attention at all compared to this IIB pp-wave. The reason for such a different treatment is that having the AdS/CFT correspondence allows for an identification of how the limit acts on the dual CFT; we therefore end up with a quantizable string theory dual to a sector of observables in the CFT. As we will carefully analyze, the situation illustrated in figure III.1 radically changes and there will be overlapping regimes of both sides where comparisons can be performed.

Let us then describe how the limit acts on both sides. On the string theory side, let us call E the energy of the stringy excitations and J their angular momentum in the first of the 2-planes of the $S^5 \subset \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$. The metric (III.10) is the limit of $AdS_5 \times S^5$ where one focuses on a null geodesic at the origin of the 2nd and 3rd of the $\mathbb{R}^2$ factors (which makes it a great circle in the S^5). The string states that survive in the limit are those with

$$J \sim R^2 \sim \sqrt{\lambda}\,, \qquad E - J = \text{fixed}\,, \qquad \text{as} \;\; R/l_s \to \infty. \tag{III.12}$$

It is important to remark that the limit $R/l_s = \lambda^{1/4} = (g_s N)^{1/4} \to \infty$ can be taken in two different ways. If we want to keep string interactions, we fix g_s and we let $N \to \infty$ keeping $J \sim N^{1/2}$. If we prefer to obtain a free string theory, which can be useful to analyze the spectrum, we can first take a conventional 't Hooft limit $g_s \to 0$ keeping λ fixed, and the we perform a large 't Hooft coupling limit $\lambda \to \infty$ keeping $J \sim \sqrt{\lambda}$.

We now use the AdS/CFT dictionary to translate this limit into the CFT. The energy was measured in global AdS coordinates and must be identified with the conformal dimension of its dual YM operator. The angular momentum is the charge under an $SO(2) \subset SO(6)$ subgroup of the isometries of the S^5, and it must be identified with the R-charge of the dual operator under a $U(1)$ subgroup of $SO(6)$. Finally, using the relation $R^4 = l_s^4 g_{YM}^2 N$ we can translate (III.12) into

$$J \sim \sqrt{\lambda}\,, \qquad \Delta - J = \text{fixed}\,, \qquad \lambda \to \infty. \tag{III.13}$$

There are two immediate points to make here.

- The first one is that the string states in the pp-wave background will be dual to operators with very large conformal dimension and R-charge. There is a BPS bound that follows from the $PSU(2,2|4)$

superalgebra[4] that implies that all operators must satisfy

$$\Delta \geq J\,, \tag{III.14}$$

with equality being valid only for 1/2-BPS operators. There are a large number of operators that saturate this bound, and they just differ by the number of traces in the adjoint of the gauge group (see the appendix A for a proper discussion on chiral operators). There is an identification in the AdS/CFT correspondence that relates n-trace operators in the YM side to n-particle states in string theory one. This correspondence is based on the observation that, as states with different number of particles are orthogonal in supergravity, they should correspond to 'orthogonal' operators in the YM side, where 'orthogonality' is to be understood as

$$\langle O_1 O_2 \rangle_{CFT} = 0\,. \tag{III.15}$$

In general, it is the case that if two operators contain a different number of traces, then $\langle O_1 O_2 \rangle \sim 1/N^a$ with $a > 0$, so that the number of traces is a good quantum number in the large N limit. However, this result starts to fail when the number of fields inside each operators start to grow due to large combinatoric factors. Operators with different number of traces start to mix and the correspondence to multiparticle states starts to fail. In our present case we are on the edge to run into this problem. Although one conventionally takes the ground state of the string in the pp-wave background to be dual to a single-trace operator

$$\text{BMN ground state } |0\rangle \;\longrightarrow\; \mathcal{O}_{1,J} = \mathrm{Tr}(X^J)\,, \tag{III.16}$$

it was realized in [71] that as soon as $J \sim N^{2/3}$ the true dual operator involves a linear combination with other[5] $O_{p,J}$ operators with $p \geq 2$. This kind of phase transition corresponds in the string theory side to the dual descriptions that a graviton state admits according to the value of its angular momentum. As J becomes comparable to $N^{2/3}$ the description is more appropriate in terms of Giant Gravitons.

The map from the free string theory spectrum to the operators (III.16) is still valid, as in the $g_s \to 0$ limit keeping $J \sim \sqrt{\lambda}$ we can make J as negligible as we want in front of N. This supports as well the

[4] We will see in section III.4.3 an explicit derivation of how to obtain this and other more general BPS bounds from the algebra.

[5] As we define in the appendix A, $O_{p,J}$ denotes an operator with J fields and p traces.

identification of the first excited modes of the BMN string with single trace operators obtained from (III.16) by adding a few other fields (called impurities). Schematically

$$a^\dagger ... a^\dagger \left|0\right\rangle \;\longrightarrow\; \mathcal{O}_{1,J} = \mathrm{Tr}(X^J D_\mu X \lambda Y) \,+\, \text{perm.}\,, \tag{III.17}$$

where *perm.* stands for permutations of the impurities inside the operator with suitably chosen coefficients [4]. The operators (III.17) are called BMN or *near* 1/2-BPS operators.

- As the BMN limit requires $\lambda \to \infty$, it may look like it renders perturbation theory in the YM side useless. However, another conspiracy here makes the work of BMN computationally useful. The point comes from taking into account combinatoric factors. It was proven that the correlation functions of BMN operators are not governed by the 't Hooft coupling but by an effective coupling λ' given by

$$\lambda' = \frac{\lambda}{J^2} = \frac{g_{YM}^2 N}{J^2}\,, \tag{III.18}$$

which is kept finite in the limit. This is one of the highlights of the BMN limit, as it allows to compute in perturbative SYM at very large λ (as far as $\lambda' < 1$) where the curvature of $AdS_5 \times S^5$ is very small and supergravity is a good approximation. Another point to have into account is that also the weight of non-planar diagrams is modified from

$$\frac{1}{N} \;\longrightarrow\; \frac{J^2}{N} \equiv g_2\,. \tag{III.19}$$

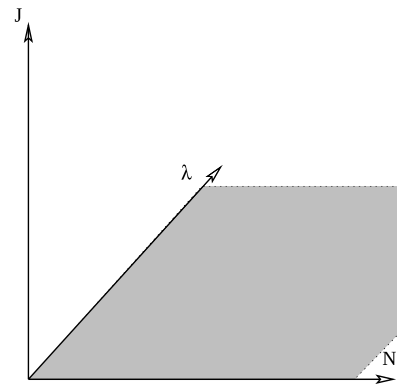
Therefore the criteria for performing perturbative field theory computations with the BMN operators change to

1. Perturbative SYM $\Longleftrightarrow\; \lambda' \ll 1 \Rightarrow \lambda \ll J^2$.
2. Perturbative + Planar SYM $\Longleftrightarrow\; \lambda \ll J^2 \ll N$.

III.2.1 Summary

We would like to visually summarize here the BMN limit. The impossibility of testing the whole AdS/CFT cannot be yet overcome, but a certain region of strings in AdS do admit comparison to a certain set of observables in the CFT. The simplification is accomplished by focusing on observables with large quantum numbers, where quantum physics typically reduce to classical physics, as we will exploit in the next sections. Loosely speaking, BMN

introduced a new axis in the (λ, N)-plane (which determined the $\mathcal{N} = 4$ SYM Lagrangian), an axis of large R-charge along which the correlation functions of some operators simplify.



(III.20)

We can now modify our plot (III.1) to incorporate J in both axes and gain a better understanding of the overlapping region opened in the BMN limit, see figure (III.2).

Yet another convenient way of understanding the BMN progress that will be useful in more complicated cases is to think of it as follows. The 1/2-BPS operators with one R-charge J have non-renormalized 2 and 3 point functions, so that they can be safely extrapolated to the string theory region. Operators a few impurities away from them *do receive* radiative corrections which can be computed in perturbative SYM and, due to the J^2 suppression, they can be extrapolated as well. Somehow, we have dug a safety tunnel in the $\{\lambda, N\}$ space; this tunnel has a nonzero radius, meaning that if we stay close to its axis (where the BPS operators live), we can still extrapolate despite the quantum corrections. It is maybe worth to illustrate this as well, as in figure III.3.

III.3 The GKP simplification

We have seen that $\mathcal{N} = 4$ SYM operators with very large $SO(2)$ R-charge are dual to a simplification of the $AdS_5 \times S^5$ background and that, furthermore, string theory is quantizable in such background. The observation of Gubser, Klebanov and Polyakov (GKP) is that one can obtain a part of the BMN results in a much simpler way which, in turn, allows for an application to many other similar cases. They proposed that some field theory operators with large quantum numbers are dual to string theory worldsheet solitons in $AdS_5 \times S^5$. Recall that the string σ-model in $AdS_5 \times S^5$ has an

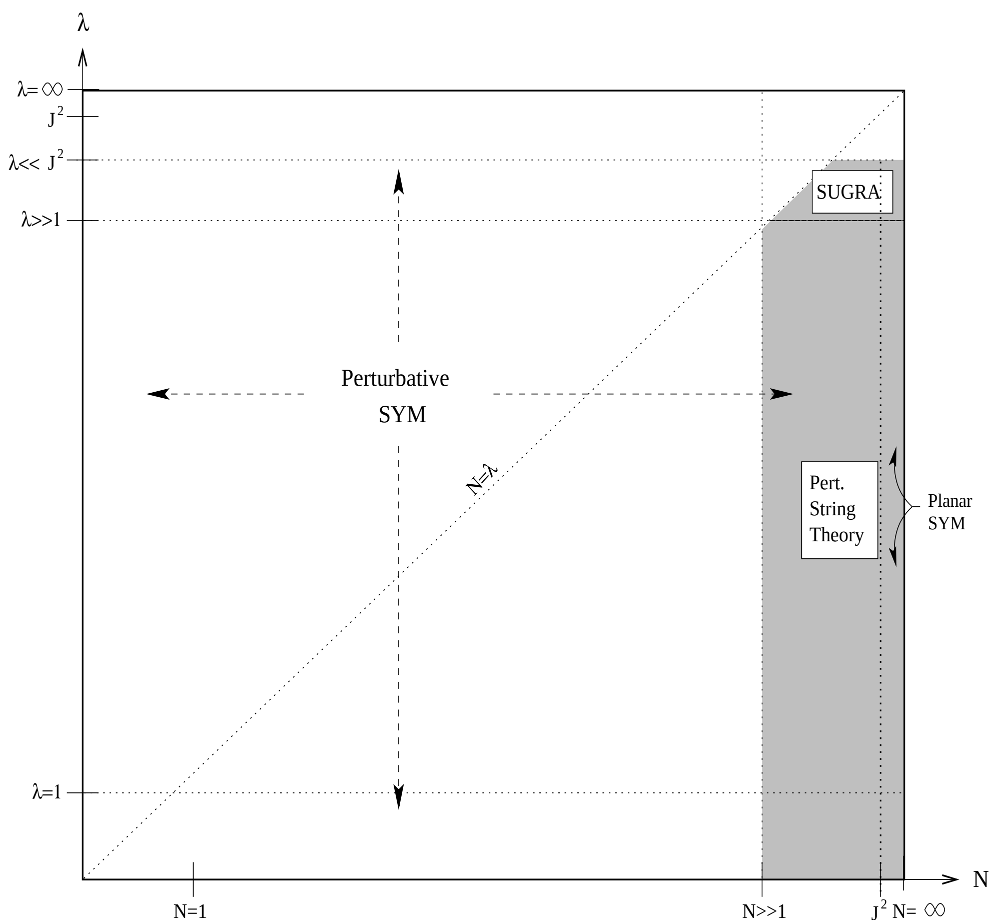


Fig. III.2: Modification of the AdS/CFT ranges of validity in the BMN sector (J is taken very large). In the shaded region there is simultaneous validity of perturbative SYM and perturbative string theory.

effective coupling $\alpha \sim 1/\sqrt{\lambda}$ as can be seen by rescaling the metric so that

$$ds^2_{AdS_5 \times S^5} = R^2 \, d\tilde{s}^2 \,, \tag{III.21}$$

which leads to a bosonic σ-model of the style

$$S_{2d} \sim \frac{1}{l_s^2} \int_{\Sigma_2} d^2\sigma \sqrt{-g_{AdS^5 \times S^5}} = \frac{R^2}{l_s^2} \int_{\Sigma_2} d^2\sigma \sqrt{-\tilde{g}} = \frac{1}{\sqrt{\lambda}} \int_{\Sigma_2} d^2\sigma \sqrt{-\tilde{g}} \,.$$

If we find a non-perturbative solution to this σ-model, then its quantum numbers (let us call them generically Q) will, by definition, depend on an inverse power of the coupling

$$Q \sim \frac{1}{\alpha^p} \sim \lambda^p \,, \qquad p > 0 \,. \tag{III.22}$$

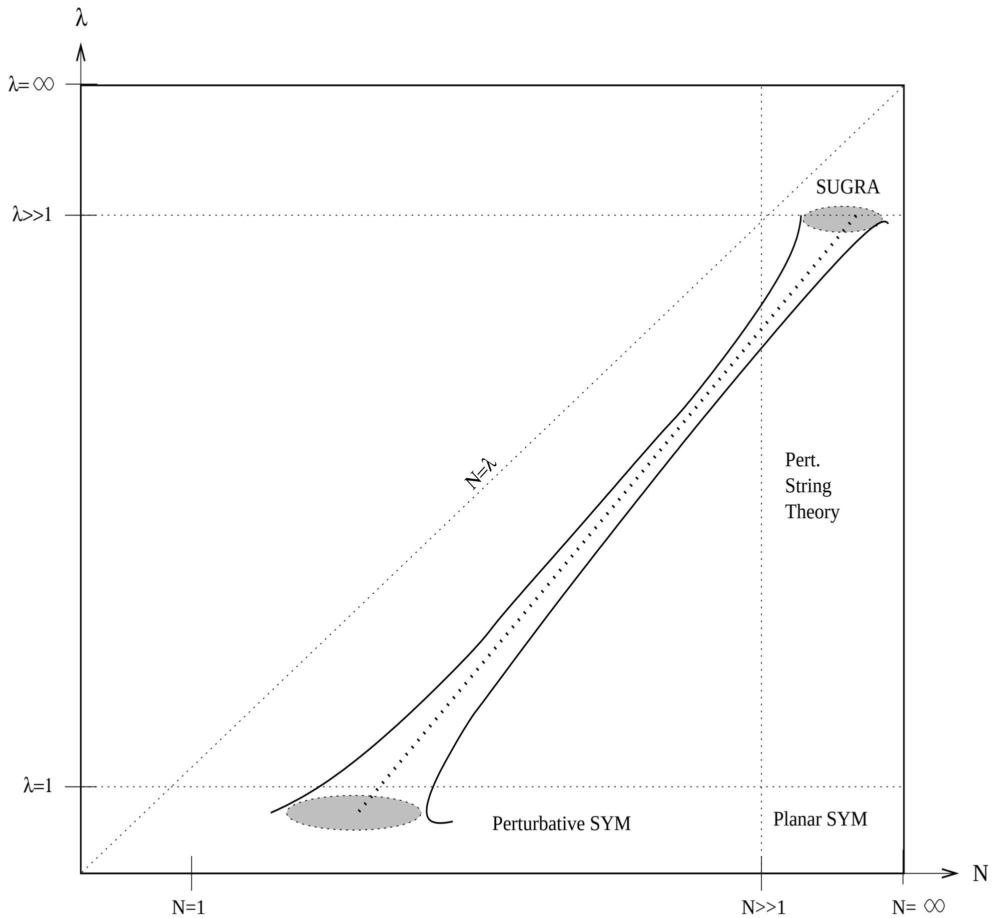


Fig. III.3: The BMN operators 'dig a safety tunnel' if we imagine placing the 1/2-BPS protected operators in its middle, and the near BPS (which receive quantum corrections) about it.

In the region of small curvatures (large λ), these charges are very large and are therefore expected to be dual to operators with a large number of fields. The expectation is that in the string side, these quantum numbers can be well approximated by their *classical* value, *i.e.* neglecting quantum corrections in α.

Note that the solitons we are talking about are not topological, in the sense that they are continuously connected to the vacuum which is typically an almost-collapsed string (the usual supergravity states). In flat space, where the full perturbative spectrum is known, one could construct these solitons by acting with creation operators on the vacuum, and we would typically obtain a coherent state in the string Hilbert space. This is not possible to do here because of the impossibility to quantize the string in $AdS_5 \times S^5$. Expanding about the classical soliton circumvents this problem.

This is a powerful proposal which, again, allows to make a lot of predictions but very few tests. The reason is as always that even though it extends the AdS/CFT dictionary, one still cannot extrapolate from SYM computations $\lambda \ll 1$ to string theory ones $\lambda \gg 1$. In other words, after computing Q in both sides, one still needs to answer the following question

- Q_{SYM} is computed at $\lambda << 1$ where Q_{string} is computed at $\lambda >> 1$. Can we extrapolate any of them to the reciprocal region in order to compare?

Let very briefly us discuss two of the most relevant applications of the GKP ideas.

III.3.1 Twist two operators

The example originally given in GKP involved the identification of twist-two operators with folded spinning strings in the AdS factor of $AdS_5 \times S^5$. These operators have $\Delta = S + 2$, where S is the charge under one of the $SO(2) \subset SO(2,4)$ and have the form

$$\mathrm{Tr}\; X \nabla_{(\mu_1} \ldots \nabla_{\mu_n)} X \,. \tag{III.23}$$

They are present in non-supersymmetric theories as well and, being non-chiral, their conformal dimension receives quantum corrections. These are all believed (even non-perturbatively) to have a leading large S contribution proportional to $\ln S$, so that

$$\Delta \underset{S \gg 1}{\approx} S + f(\lambda) \ln S \,. \tag{III.24}$$

If $f(\lambda)$ is computed in perturbation theory, then one gets a power series in λ and its first term (for QCD) can be checked experimentally!

The identification of GKP with the spinning string allows for a simple computation of the charges of this soliton and the result was

$$\Delta \underset{S\gg 1}{\approx} S + \frac{\sqrt{\lambda}}{\pi} \ln(S/\sqrt{\lambda})\,. \tag{III.25}$$

As this is a stringy result, it must be understood as valid for very large λ, so that it *predicts* the behavior of $f(\lambda)$ away from perturbation theory. This is another example of a prediction. There is no hope for comparison since, on the one hand, the SYM operators are not protected nor their effective coupling is combinatorially suppressed and, on the other hand, quantum σ-model corrections to the string classical values would become more and more relevant as we move away from large λ.

III.3.2 BMN operators

We apply the GKP ideas to the BMN ground state operator (III.16). As it only carries nonzero Δ and J, we must look for a string state with rotation only in the S^5 factor of $AdS_5 \times S^5$, which turns out to be an almost collapsed closed string. We will see in detail in the next section how these ideas are carried out in similar but more sophisticated cases, so here we just cite the results for the BMN operators. The classical string approximation yields the relation $E = J$ which is the exact relation $\Delta = J$ for the 1/2-BPS operator. Remarkably, the first σ-model correction to this energy is able to reproduce the whole spectrum of the string in the pp-wave background! So the diagram III.4 holds.

This case is essentially different from the previous one (where the string rotated in the AdS factor) because both the ground state operator and its dual string solution preserved 1/2 supersymmetry. This has helped to support the fact that the answers to the questions we posed are positive: yes, $Q_{string} = Q_{classical}$ at large J; and yes, Q_{CFT} can be extrapolated to the stringy region. Thus supersymmetry seems to be in the heart of this test of AdS/CFT.

III.4 Trying to check AdS/CFT beyond supersymmetry

We have seen that the progress initiated by BMN and then improved by GKP is able to provide a check of the AdS/CFT in a near-BPS sector of operators/states. All other quantitative work must be interpreted as

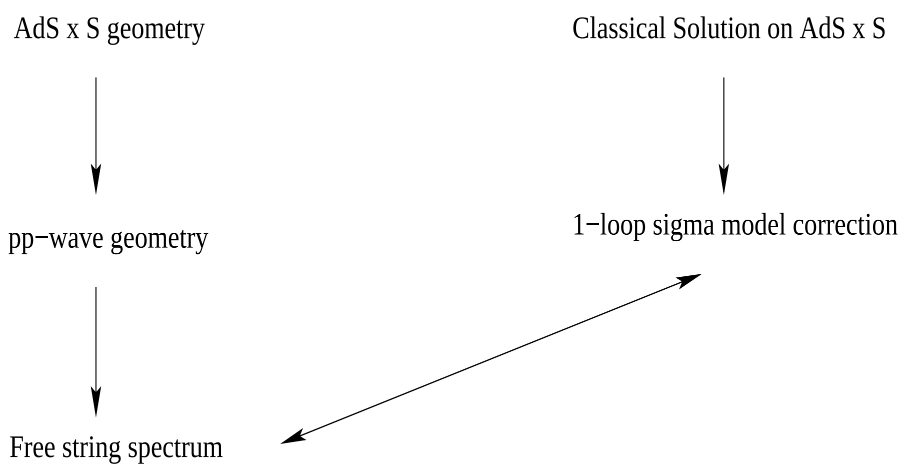


Fig. III.4: The shortcut provided by GKP.

providing predictions. One might suspect that what is behind the tests described above is supersymmetry rather than AdS/CFT, although after 7 years of duality most people believe in the strongest possible version of the AdS/CFT.

So it remains as a major challenge to be able to test (or reject!) the AdS/CFT conjecture away from supersymmetric sectors. There has been some recent progress along these lines by an intensive exploitation of the GKP ideas. Essentially, one would like to follow this route:

1. Start with a SYM operator with large quantum numbers that is far from any BPS one.

2. Compute a radiative correction, say, to the conformal dimension.

3. Identify the string soliton dual to this operator.

4. Compute the classical energy and relate it to its classical charge.

5. See if both computations can be compared.

This task will most of the times fail in the last point, the twist-two operators we discussed above being an example. As the operators we want to work with are far from BPS, the dual string states will be far from supersymmetric ones, and it is then very difficult to answer positively to

1. is the classical solution stable at all?

2. can we neglect quantum σ-model corrections to $E_{classical}$?

3. are the perturbative corrections to Δ_{CFT} suppressed by $1/J^2$ factors, so that they can be extrapolated?

III.4.1 Rotating strings in spheres

We will now describe the attempts to carry on this enterprize which, despite the difficulties mentioned above, started around April 2003. These recent results are based on the obtention of new σ-model solitons. Let us start from the most basic point: which kind of solutions should we look for? The BMN ground state operator carries only nonzero (Δ, J) charges[6] and it is dual to an almost-particle (a collapsed string) travelling on S^5. If we want to succeed with the identification of operators/states we had better look for other kind of operators with definite charges. The simplest possibility is maybe an operator with (Δ, S) charges, but this is similar to the twist-two ones and we saw that they do not allow for comparisons. Next attempt: consider (Δ, S, J); it was seen that they fail again [72]. Next attempt: consider (Δ, J_1, J_2). This possibility is distinguished from the previous ones as it must correspond to an extended (as opposite to collapsed) string rotating in the sphere; a particle could never carry two independent angular momenta in a sphere. This means that it must correspond to a truly *stringy* state and if the correspondence worked, it would be a test *away from supergravity*.

A string solution with two angular momenta on a sphere had been long known [73] and its relevance for AdS/CFT was first introduced by Frolov and Tseytlin (FT) in [7]. A good way to understand them may start with understanding why are they difficult to find. An extended object made of mass (so that each of its points attracts the others) finds it very hard to stabilize in our familiar $\mathbb{R}^3$ only by rotating. This is because the rotation of a rigid body is always about one axis (which may rotate as well). Two points along the axis do not feel a centrifugal force, so that they just tend to approach by gravitation attraction. If the body was not rigid, it would just collapse. Flat galaxies are an example; they are more stable by concentrating all their mass in a plane perpendicular to the axis of rotation.

The situation is dramatically different in $\mathbb{R}^4$ because $SO(4)$ is a group of rank 2. This means that a body can have two independent angular momenta

[6] We recall that we use the convention $(\Delta, S_1, S_2, J_1, J_2, J_3)$ to label the charges in the Cartan subalgebra $SO(1,1) \times SO(2)^5$ of $SO(2,4) \times SO(6)$. The name *spin* refers to S-charges and corresponds to motion in AdS_5. The name *angular momentum* refers to J-charges and corresponds to motion in the S^5.

along two different planes. It turns out that this mechanism can be used to provide the same centrifugal force to each point of a string, so that gravity is compensated everywhere. This is achieved if we take the two angular momenta to be equal $J_1 = J_2 = J$.

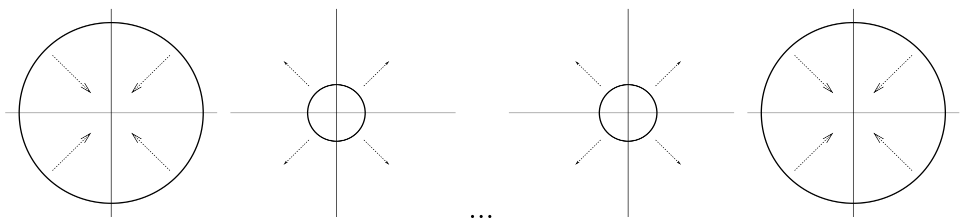


Fig. III.5: On the left, the projections of the string in the 12 and 34 planes of $\mathbb{R}^4$ at fixed t_0; the string contracts in the first and grows in the latter. On the right, the same but a bit later; the situation is reversed. Remark: unlike in $\mathbb{R}^3$, all points suffer centrifugal acceleration.

The work of [73] actually included similar solutions for general n-dimensional relativistic surfaces, which could be constructed as long as their ambient space was large enough to allow for enough independent angular momenta. A remarkable property of these solutions is that they all actually happened to lie at all times on a sphere rather than in the whole ambient space. For example, the string just described happened sweep an $S^3 \subset \mathbb{R}^4$ rather than the whole $\mathbb{R}^4$. Thus these surfaces are ready to play a game in AdS/CFT by placing them in the sphere factors of $AdS_p \times S^q$. This is what was done in [7], where they constructed a rotating string in the S^5 factor of $AdS_5 \times S^5$ with two equal $SO(6)$ momenta.

However, we encounter here the first of the obstacles we mentioned above. Whereas it was proven [74] that all the solutions in flat space of [73] are always stable, it was proven that the ones in S^5 are always unstable [7, 8]. Both results are not in contradiction because stability is a property of second order fluctuations about a given solution, so that it probes the curvature of the background space.

Should we just stop here our enterprize? Let us postpone the issue of stability and try to proceed, as everyone has done in the series of papers that have appeared after the work of Frolov and Tseytlin [9, 10, 11, 12, 13, 14, 15]. We now want to obtain the classical energy as a function of its classical charges. For that we will need to describe the solution in more detail.

III.4.2 Strings with 3 angular momenta

Here we shall describe in detail the embedding of the strings we discussed above and obtain the relation among their charges. Note however that the simplest string we described lives only in an $S^3 \subset S^5$, so that it still has room for another angular momentum describing the motion of the S^3 inside the S^5. So we will use the notation

- $(\omega, J') \leftrightarrow$ angular velocity and momentum due to 'self rotation' in the two planes of S^3,
- $(\nu, J) \leftrightarrow$ angular velocity and momentum due to the S^3 motion in S^5.

Note that both types of motion are of very different physical origin.

Note: In the following sections we restrict our attention to strings with two of the three angular momenta being equal. Although solutions with 3 independent angular momenta were found afterwards, we prefer to stick to the less general case because it follows from the intuition we described above. The generalization to 3 independent J's does not provide any further insight and will be considered in the appendix B

We will be considering solutions of the classical equations of motion derived from the Nambu-Goto Lagrangian

$$\mathcal{L} = -\frac{1}{\alpha'}\sqrt{-\det g}\,, \tag{III.26}$$

where g is the worldsheet metric induced from the $AdS_5 \times S^5$ spacetime metric. The strings we want to consider lie at the origin of AdS_5 and rotate only in the S^5, so effectively they live in a $\mathbb{R} \times S^5$ subspace of $AdS_5 \times S^5$ with metric

$$ds^2 = R^2\left(-dt^2 + d\Omega_5^2\right)\,, \tag{III.27}$$

where $d\Omega_5^2$ is the $SO(6)$-invariant metric on the unit five-sphere and we recall that $R^2 = \alpha'\sqrt{\lambda}$. The S^5 may be viewed as the submanifold $|\mathbf{W}| = 1$ of $\mathbb{R}^6$ with Cartesian coordinates W_i $(i = 1, \ldots, 6)$. For the parametrization with

$$\begin{aligned} W_1 + iW_2 &= \cos\theta\, e^{i\chi}\,, \\ W_3 + iW_4 &= \sin\theta\,\cos\phi\, e^{i\alpha}\,, \\ W_5 + iW_6 &= \sin\theta\,\sin\phi\, e^{i\beta}\,, \end{aligned} \tag{III.28}$$

this gives

$$d\Omega_5^2 = d\theta^2 + \sin^2\theta\, d\phi^2 + \cos^2\theta\, d\chi^2 + \sin^2\theta \cos^2\phi\, d\alpha^2 + \sin^2\theta \sin^2\phi\, d\beta^2 \,. \tag{III.29}$$

We fix the worldvolume reparametrization invariance by the gauge choice

$$t = \tau\,, \qquad \alpha = \sigma\,, \tag{III.30}$$

where τ and σ are the worldsheet coordinates.[7] The string solutions of interest correspond to circular, rotating strings supported against collapse by their angular momenta; they are given by

$$\theta = \theta_0\,, \qquad \chi = \nu\tau\,, \qquad \phi = \omega\tau\,, \qquad \beta = \sigma\,, \tag{III.31}$$

where θ_0 is a constant in the interval $[0, \pi/2]$. The intuitive picture discussed above is realized here in the fact that, at any instant, the string is a circle in a two-plane contained within the 3456-space; this plane rotates with angular velocity ω. In turn, the string's center of mass rotates with angular velocity ν around a circle in the 12-plane.

Under the above circumstances, the Nambu-Goto equations are solved if either of the following relations hold[8]

$$\begin{array}{lll} (i) & \cos\theta_0 = 0\,, \qquad \omega^2 < 1\,, & \\ (ii) & \cos 2\theta_0 = \dfrac{\omega^2 - 1}{\omega^2 - \nu^2}\,, \qquad \nu^2 < 1\,, \qquad 2\omega^2 - \nu^2 - 1 > 0\,. \end{array}$$

The restrictions on the angular velocities follow from demanding the reality of both $\mathcal{L}$ and θ_0. The energy of the rotating string is

$$E = \sqrt{\lambda}\, |\sin\theta_0|\, \Delta^{-1/2}\,, \qquad \Delta \equiv 1 - \nu^2 \cos^2\theta_0 - \omega^2 \sin^2\theta_0\,, \tag{III.32}$$

while the only non-zero components of the angular momentum two-form, J_{ij}, in the Cartesian coordinates W_i, are

$$J \equiv J_{12} = E\,\nu\, \cos^2\theta_0\,, \qquad J' \equiv J_{35} = J_{46} = \frac{1}{2}\, E\,\omega\, \sin^2\theta_0\,. \tag{III.33}$$

[7] Note that in our conventions both the spacetime and the worldsheet coordinates are dimensionless, which implies that the energy of the string, E, is also dimensionless. The corresponding dimensionful time and energy are obtained by rescaling the dimensionless ones by appropriate powers of R.

[8] Note that in case (i) the string always lies at the origin of the 12-plane, which is a singular submanifold of the coordinate system we are using; ν is the angular velocity in a circle of zero radius. In practice this does not cause a problem because ν is irrelevant in this case.

Thus, J and $2J'$ are the momenta conjugate to χ and ϕ, respectively. Their values for the two possible solutions of the equations of motion are

$$
\begin{aligned}
(i) \qquad & E = \frac{\sqrt{\lambda}}{\sqrt{1-\omega^2}}\,, \qquad J = 0\,, \qquad J' = \frac{\sqrt{\lambda}\,\omega}{\sqrt{1-\omega^2}}\,, \qquad \text{(III.34)}\\
(ii) \qquad & E = \frac{\sqrt{\lambda}}{\sqrt{\omega^2-\nu^2}}, \;\; J = \sqrt{\lambda}\,\frac{(2\omega^2-\nu^2-1)\,\nu}{2\,(\omega^2-\nu^2)^{3/2}}, \;\; J' = \sqrt{\lambda}\,\frac{\omega\,(1-\nu^2)}{2\,(\omega^2-\nu^2)^{3/2}}\,.
\end{aligned}
$$

In the first case it is easy to express the energy solely as a function of the angular momentum, with the result

$$
(i) \qquad E = \sqrt{(2J')^2+\lambda} = |2J'|\left[1+\mathcal{O}\left(\frac{\lambda}{J'^2}\right)\right]. \qquad \text{(III.35)}
$$

The expression for E in terms of J and J' can also be found explicitly in the second case, but the result is rather messy. However, when expanded for large J, J', it yields

$$
(ii) \qquad E = (|J|+|2J'|)\left[1+\mathcal{O}\left(\frac{\lambda}{J^2},\frac{\lambda}{J'^2}\right)\right]. \qquad \text{(III.36)}
$$

Summary:

The last two equations is what we were looking for. They provide the string classical result for the relation between the energy and the angular momentum. There are a few comments to make here:

1. Both expressions can be expanded for $J^2 \gg \lambda$ in such a way that only integer powers of λ/J^2 appear. This did not happen for the twist-two operators. Had not it been this way, there would have been no hope to compare with a perturbative field theory result.

2. The first term, which is λ independent, is precisely the classical dimensions that the following operators have

$$
\begin{aligned}
\mathcal{O}(J') &= \mathrm{Tr}\left(Y^{J'}Z^{J'}\right) + \text{perm.}\,, \qquad \text{if } J=0\,,\\
\mathcal{O}(J,J') &= \mathrm{Tr}\left(X^{J}Y^{J'}Z^{J'}\right) + \text{perm.}\,, \qquad \text{if } J\neq 0\,. \qquad \text{(III.37)}
\end{aligned}
$$

   So it is tempting to relate our string solutions to these operators.

3. The proposed operators are non-chiral, which agrees with the fact that these solutions are non-supersymmetric. Indeed, they are a huge number of impurities away from the BMN ground state operator. It

is tempting to compare the first terms in the expressions for $E(J,J')$ with a perturbative SYM computation of the anomalous dimensions of the corresponding operators. As $E(J,J')$ is a power series in λ/J^2, it seems reasonable to suspect that the SYM corrections will also be controlled by $\lambda' = \lambda/J^2$ as it happened for BMN operators.

Everything leads to the conclusion that it is worth doing the computation in the SYM side. This process has been carried out in a number of papers [9, 10, 11, 12, 13, 14] and the success has been spectacular. The SYM computation has been possible mainly due to the realization that one can map the problem at 1-loop to different types of spin chains.

We will come back to the discussion of the SYM computation later. Now, we will spend some time trying to understand what is actually going on in the string side. Recall that the string solution is not even stable when $J < 2J'/3$ so when Tseytlin and Frolov computed the 1-loop σ-model correction to it in this regime, they were actually doing it *about a tachyonic vacuum!* They used the 1-loop result to show that it is negligible against the classical one for $J, J' \gg 1$. Then arguments were given that negligibility holds to all orders in σ-model corrections analogously to what was argued for the BMN case.

Altogether it seems like the comparison is being too successful. The proposal that we gave in [37, 38] is that actually we are testing a near BPS sector again. The first indications are:

1. All the perturbations that develop tachyonic masses (and are therefore responsible for instabilities) have the property that these masses vanish in the large angular momenta $J, J' \gg \sqrt{\lambda}$ limit.[9] This means that the string becomes asymptotically marginally stable.

2. The leading terms in the energy saturate bounds that can be derived from the background superalgebra in the large angular momenta limit. The general bound leads to preservation of 1/8-supersymmetry; this is enhanced to 1/4 for the case $J = 0$.

3. The last and definitive proof is given below (section III.4.4) where we confirm by standard k-symmetry arguments that the solutions preserve the mentioned fractions of supersymmetry in the limit. In turn, this resolves the ambiguity posed by the asymptotic marginal stabil-

[9] The relevant tachyonic masses appear in formulas (4.34) of [7] and (2.35), (2.36) of [8].

ity: the solution becomes stable in the limit as it carries the minimum energy for the given charges.

We now proceed to show how the BPS bound arises from the superalgebra and then we proceed to the κ-symmetry analysis. A careful discussion of the physical meaning of the limit considered here is given afterwards in section III.4.5.

III.4.3 BPS Bound from the Superalgebra

The energy of any supersymmetric state in $AdS_5 \times S^5$ must saturate a BPS bound that follows from the $PSU(2,2|4)$ isometry superalgebra of the $AdS_5 \times S^5$ vacuum, and hence its energy can be expressed as a function of their charges alone. In this section we review the BPS bound for states that carry the same type of charges as the rotating, circular strings above, that is, energy and angular momenta on the S^5. The BPS bound we will derive may be equally well understood as a statement about the supersymmetry properties of operators in the dual CFT as we will discuss later.

Let γ_m ($m = 0, \ldots, 4$) be the 4×4 five-dimensional Dirac matrices for AdS_5 and let Q^i be the four AdS_5 Dirac spinor charges, transforming as the **4** of $SU(4)$. The non-zero anticommutators are

$$\{Q^i, Q^\dagger_j\} = \gamma^0 \left[\left(\gamma_m P^m + \frac{1}{2}\gamma_{mn} M^{mn} \right) \delta^i{}_j + 2\mathbb{I}\, B^i{}_j \right] , \qquad \text{(III.38)}$$

where P, M are the AdS_5 charges and B is the hermitian traceless matrix of $SU(4)$ charges. For our spinning string configurations the only non-zero AdS charge is the energy $P^0 = E$; in this case (III.38) reduces to

$$\{Q^i, Q^\dagger_j\} = \mathbb{I}\, \delta^i{}_j E + 2\gamma^0 B^i{}_j \, . \qquad \text{(III.39)}$$

By means of an $SU(4)$ transformation we may bring B to diagonal form with diagonal entries b_i ($i = 1, 2, 3, 4$) satisfying

$$b_1 + b_2 + b_3 + b_4 = 0. \qquad \text{(III.40)}$$

The eigenvalues of the 16×16 matrix $\{Q, Q^\dagger\}$ are therefore $E \pm 2b_1, E \pm b_2, E \pm b_3, E \pm b_4$, each being doubly degenerate. Since this matrix is manifestly positive in any unitary representation, unitary implies the bound

$$E \geq 2b \qquad b = \sup\{|b_1|, |b_2|, |b_3|, |b_4|\} \, . \qquad \text{(III.41)}$$

When the bound is saturated the matrix $\{Q, Q^\dagger\}$ will have zero eigenvalues; the possible multiplicities are $2, 4, 8, 16$. The maximum number (16) occurs when $b_i = b$ for all i, in which case (III.40) implies $b = 0$ and hence $E = 0$; this is the adS_5 vacuum. Otherwise, one has preservation of $1/8, 1/4, 1/2$ supersymmetry when $\{Q, Q^\dagger\}$ has $2, 4, 8$ zero eigenvalues, respectively.

The eigenvalues of B are $SU(4)$ invariants and hence determined in any $SU(4)$ irrep by that irrep's Dynkin labels (d_1, d_2, d_3). Conversely, the Dynkin labels are determined by the eigenvalues of B, and consideration of the highest weight state leads to the relation

$$d_1 = b_1 - b_2, \qquad d_2 = b_2 - b_3, \qquad d_3 = b_3 - b_4. \tag{III.42}$$

Given the constraint (III.40), this can be inverted to give

$$\begin{aligned} b_1 &= \frac{1}{4}(3d_1 + 2d_2 + d_3)\ , \\ b_2 &= \frac{1}{4}(-d_1 + 2d_2 + d_3)\ , \\ b_3 &= \frac{1}{4}(-d_1 - 2d_2 + d_3)\ , \\ b_4 &= \frac{1}{4}(-d_1 - 2d_2 - 3d_3)\ . \end{aligned} \tag{III.43}$$

The $SU(4)$ charges of the spinning strings considered here correspond to irreps with Dynkin labels [7]

$$[d_1, d_2, d_3] = [J' - J, 0, J + J'] \qquad \text{if} \qquad J' > J\,, \tag{III.44}$$

$$[d_1, d_2, d_3] = [0, J - J', 2J'] \qquad \text{if} \qquad J' \leq J\,. \tag{III.45}$$

It follows, for either case, that the four (unordered) eigenvalues of B are

$$\{J' - \frac{1}{2}J\,,\, \frac{1}{2}J\,,\, \frac{1}{2}J\,,\, -J' - \frac{1}{2}J\}\,. \tag{III.46}$$

Using this in (III.41), we deduce that

$$E \geq |J| + 2|J'|. \tag{III.47}$$

When this bound is saturated the matrix of anticommutators of supersymmetry charges will have zero eigenvalues, corresponding to the preservation of some fraction of supersymmetry. Let us determine this fraction under the assumption that $J' > J$. In this case the Dynkin labels are given by (III.44) and hence

$$b_1 = J' - \frac{1}{2}J, \qquad b_2 = \frac{1}{2}J, \qquad b_3 = \frac{1}{2}J, \qquad b_4 = -J' - \frac{1}{2}J. \tag{III.48}$$

Generically, $|b_1| = b$ and all other eigenvalues of B have absolute value less than b so the supersymmetry fraction is $1/8$. However, this fraction is enhanced to $1/4$ if $J = 0$ because then $|b_1| = |b_4| = b$ with $|b_2|, |b_3| < b$.

A similar analysis for $J \geq J'$ again yields the fraction $1/8$ generically, with enhancement to $1/2$ if $J' = 0$; in this case the string reduces to a point-like string orbiting the S^5 along an equator, as considered by Berenstein, Maldacena and Nastase (BMN) [4]. Finally, if $J = J' = 0$ then $b_i = 0$ for all i, $E = 0$, and all supersymmetries are preserved, as expected for the $AdS_5 \times S^5$ vacuum.

Redoing the computation for strings with 3 independent angular momenta we find that the bound is $E \geq |J_1| + |J_2| + |J_3|$, with 1/2-, 1/4- or 1/8- preservation of supersymmetry for one, two or three nonzero angular momenta respectively.

III.4.4 Supersymmetry from κ-symmetry

The supersymmetries preserved by a IIB string correspond to complex Killing spinors ϵ of the background that satisfy[10]

$$\Upsilon\,\epsilon = \sqrt{-\det g}\,\epsilon\,, \qquad \Upsilon = X'^M \dot{X}^N \gamma_{MN}\,K\,, \tag{III.49}$$

where K is the operator of complex conjugation, and γ_M are the (spacetime-dependent) Dirac matrices.

Recall that we are interested in strings that live in the $\mathbb{R}\times S^5$ submanifold of $AdS_5 \times S^5$ with metric (III.29), and whose embedding is specified by equation (III.31). Under these circumstances[11]

$$\sqrt{-\det g} = \sin\theta\sqrt{1-a^2-b^2}\,, \qquad \dot{X}\cdot\gamma = \Gamma_t + a\,\Gamma_\chi + b\,\Gamma_\phi\,, \tag{III.50}$$

where $\Gamma_\theta, \Gamma_\phi, \ldots$ are ten-dimensional tangent space (*i.e.* constant) Dirac matrices, and

$$a = \nu\,\cos\theta\,, \qquad b = \omega\,\sin\theta\,. \tag{III.51}$$

Note that the Lorentzian signature of the induced worldsheet metric implies that

$$a^2 + b^2 \leq 1\,. \tag{III.52}$$

[10] Thus, $\Upsilon/\sqrt{-\det g}$ is the matrix Γ_κ appearing in the kappa-symmetry transformation of the fermionic variables of the Green-Schwarz IIB superstring.

[11] As the radius R cancels in the final result we set $R = 1$ in this section.

The Killing spinors of $AdS_5 \times S^5$ restricted to the relevant submanifold take the form

$$\epsilon = e^{\frac{t}{2} i \tilde{\Gamma}} \, e^{\frac{\theta}{2} i\gamma_* \Gamma_\theta} \, e^{\frac{\phi}{2} \Gamma_{\theta\phi}} \, e^{\frac{\chi}{2} i\gamma \Gamma_\chi} \, e^{\frac{\alpha}{2} \Gamma_{\theta\alpha}} \, e^{\frac{\beta}{2} \Gamma_{\phi\beta}} \, \epsilon_0 \,, \tag{III.53}$$

where ϵ_0 is a constant spinor, $\gamma_* = \Gamma_{\theta\phi\chi\alpha\beta}$, and $\tilde{\Gamma}$ is a constant matrix that commutes with all other matrices above (its specific form will not be needed). In our conventions all these matrices are real. For our configurations $\dot{X} \cdot X' = 0$, so the supersymmetry preservation condition (III.49) can be written as

$$(X' \cdot \gamma) \left(\dot{X} \cdot \gamma \right) \epsilon = -\sqrt{-\det g} \; K \, \epsilon \,. \tag{III.54}$$

This must be satisfied for all τ, σ, but it is useful to first consider $\tau = 0$, in which case it reduces to

$$\sin\theta \left(\dot{X} \cdot \gamma \right) \, \Gamma_\alpha \, K \, e^{\frac{\theta}{2} i\gamma_* \Gamma_\theta} \, e^{\frac{\sigma}{2} \left(\Gamma_{\theta\alpha} + \Gamma_{\phi\beta} \right)} \, \epsilon_0 = \sqrt{-\det g} \, e^{\frac{\theta}{2} i\gamma_* \Gamma_\theta} \, e^{\frac{\sigma}{2} \left(\Gamma_{\theta\alpha} + \Gamma_{\phi\beta} \right)} \, \epsilon_0 \,. \tag{III.55}$$

It can be shown that in order for this equation to be satisfied for all σ, one must impose

$$\Gamma_{\theta\alpha\phi\beta} \, \epsilon_0 = \, \epsilon_0 \,, \tag{III.56}$$

in which case the equation becomes

$$\left[\Gamma_t + (\cos\theta - \sin\theta \, i\gamma_* \Gamma_\theta) \, (a \, \Gamma_\chi + b \, \Gamma_\phi) \right] \, \Gamma_\alpha \, K \, \epsilon_0 = \sqrt{1 - a^2 - b^2} \, \epsilon_0 \,. \tag{III.57}$$

Equations (III.56) and (III.57) are equivalent to the two equations

$$A \, \epsilon_0 \;=\; \epsilon_0 \,, \qquad A \equiv a \, \cos\theta \, \Gamma_{t\chi} + b \, \sin\theta \, i\Gamma_{t\chi\alpha\beta} \,, \tag{III.58}$$

$$B \, K \epsilon_0 \;=\; \sqrt{1 - a^2 - b^2} \, \epsilon_0 \,, \qquad B \equiv a \, \sin\theta \, i\Gamma_{\phi\beta} + b \, \cos\theta \, \Gamma_{\phi\alpha} \tag{III.59}$$

Given that a and b are non-zero, it follows from (C-2) that

$$i\Gamma_{\alpha\beta} \, \epsilon_0 = s \, \epsilon_0 \,, \qquad \Gamma_{t\chi} \, \epsilon_0 = s \, \epsilon_0 \,, \tag{III.60}$$

and

$$a \, \cos\theta + s \, b \, \sin\theta = \tilde{s} \,, \tag{III.61}$$

where s and $\tilde{s}$ are independent signs. The latter relation is compatible with the restriction (III.52) if and only if $b \cos\theta = s \, a \, \sin\theta$, and these two relations for a and b imply, given (III.51), that

$$\nu = \tilde{s} \,, \qquad \omega = s \, \tilde{s} \,. \tag{III.62}$$

It then follows that the equation (III.59) is trivially satisfied, and that $\sqrt{-\det g} = 0$. The string worldsheet must therefore be null, which is only

possible for a tensionless string. Although the IIB superstring is not tensionless, the energy due to the rotation is much greater than the energy due to the tension in the limit of large angular momentum. So in this limit the string is effectively tensionless.

We continue now by considering only the tensionless string, for which the supersymmetry preserving condition (III.54) reduces to

$$\left(\dot{X} \cdot \gamma\right) \epsilon = 0 \,. \tag{III.63}$$

The analysis of this equation for $\tau = 0$ reproduces the results already obtained from an analysis of (III.55), which are summarized by the projections

$$\Gamma_{\theta\alpha\phi\beta}\, \epsilon_0 = \epsilon_0 \,, \qquad \Gamma_{t\chi}\, \epsilon_0 = \omega\nu\, \epsilon_0 \,, \qquad i\Gamma_{\alpha\beta}\, \epsilon_0 = \omega\nu\, \epsilon_0 \,. \tag{III.64}$$

It is now straightforward to check that a spinor ϵ_0 satisfying these conditions solves (III.63) for all τ and σ. It thus follows that the generic null FT string preserves 1/8 of the 32 supersymmetries of the IIB $AdS_5 \times S^5$ vacuum.

We have assumed above that a and b are non-zero. A solution with $b = 0$ has $J' = 0$ and corresponds to a point-like, collapsed string moving along a great circle of S^5, as considered in [4], whereas a solution with $a = 0$ has $J = 0$. Redoing the analysis it is easy to see the former preserve 1/2 of the supersymmetry. Similarly, in the second case one finds that the necessary and sufficient conditions for preservation of supersymmetry are

$$\omega^2 = 1 \,, \qquad \cos\theta_0 = 0, \tag{III.65}$$

and that the projections on ϵ are

$$\Gamma_{\theta\alpha\phi\beta}\, \epsilon_0 \ = \ \epsilon_0 \,, \qquad i\Gamma_{t\chi\alpha\beta}\epsilon_0 = \omega\epsilon_0 \,. \tag{III.66}$$

These projections preserve 1/4 of the thirty-two supersymmetries of the IIB $AdS_5 \times S^5$ vacuum.

Note: In the appendix B we consider the case of strings carrying 3 different J's directly in the tensionless case, where the steps are largely simplified.

III.4.5 Physics of the large angular momentum limit

It is worth pausing for some time to examine the results just given. It turns out that in the limit where the comparison with field theory was being made ($J^2 \gg \lambda$) the strings become null. There is a way to understand the inevitability of this. As we have seen,

- the effective string tension in an $AdS_5 \times S^5$ background is $T \sim \sqrt{\lambda}$,

- its kinetic energy K is purely due to rotation, so that $K \sim J$.

On the other hand, the expansion parameter on the field theory side had better be λ/J^p with $p > 0$ if we ever want to extrapolate SYM results to the stringy region. The conclusion is that the effective field theory coupling constant is precisely

$$\lambda' \sim \frac{\text{tension}}{(\text{kinetic energy})^p}\,, \tag{III.67}$$

and this had better be small to believe SYM results. In other words, any chance to test AdS/CFT by these methods will inevitably require a string solution whose kinetic energy is much larger than its tension: an almost tensionless string. This happens as well, of course, for the collapsed string solution dual to the BMN ground state operator.

The understanding of this issue is useful to explain what is the inherent difference between the apparently so similar solutions in which the strings rotate in AdS_5 or in S^5. Strings start growing in size as we increase the angular momentum, so that not only the kinetic energy grows but also the energy due to the tension (which is proportional to the length). If we were in flat space, both contributions remain of the same order no matter how large we take J; the string never becomes tensionless there. The same applies for strings in AdS_5 partly because it is a non-compact space.

In the S^5, however, the string soon reaches a maximum size as it can not grow larger than the S^5. At this point something highly non-trivial happens because the string is able to absorb more and more angular momenta by rotating faster, without changing its length. This is not the case, for instance, of giant gravitons; they have an upper bound on J beyond which the solution simply does not exist. In our case, the phenomenon of absorbing angular momenta keeping the length fixed finally turns the string into an almost tensionless one.

The nullity and the supersymmetry properties of these strings might be intimately related, as it is the case for particles (where susy implies $p_\mu p^\mu = 0$). We would like to mention that there have appeared a number of generalizations of the strings presented here. In the appendix B we will show that they all are actually supersymmetric in the same limit in which they become null, which is also the limit in which the comparison to field theory is made. We keep for the discussion the only apparent exception to this rule.

III.4.6 Nearly-BPS Operators

As we mentioned earlier, the macroscopic strings considered here have been proposed to be dual to operators of the form

$$\mathcal{O}(J,J') \,=\, \mathrm{Tr}\left(X^J Y^{J'} Z^{J'}\right) + \cdots \,. \tag{III.68}$$

Note that these are single-trace operators; their association to single-string states was discussed in section III.2 to be valid for $J, J' \ll \sqrt{N}$.[12] Evidence for this correspondence is that the anomalous dimensions of the $\mathcal{O}$-type operators have been computed by spin-chain methods in the one-loop planar approximation [9], and perfect agreement has been found with the string prediction in the large angular momenta limit [7, 8, 10][13]. Note that the spin chain computation implicitly assumes that $J, J' \ll \sqrt{N}$, because this condition is needed to justify the restriction to planar diagrams; as in the BMN case, non-planar diagrams are expected to be suppressed by powers of J^2/N, J'^2/N.

Our results concerning the supersymmetry of the rotating strings dual to the $\mathcal{O}$-type operators in the limit of large angular momenta imply that these operators are 'nearly-BPS' in this limit, in a sense that we now aim to clarify. Note that these operators are primary (after diagonalization of the matrix of anomalous dimensions) but not superconformal primary; for example, the operator with $J = 0$, $J' = 2$ is a descendant of the Konishi operator. Moreover, they are not 1/4-BPS or 1/8-BPS operators, since in $\mathcal{N} = 4$ SYM such BPS operators are linear combinations of multi-trace operators that involve at least[14] a double-trace or a triple-trace operator, respectively [76, 77, 78]. Therefore, although $\mathcal{O}$ is a nearly-BPS operator in the sense that its conformal dimension almost saturates a BPS bound when $\lambda/J^2, \lambda/J'^2 \gg 0$, it is not the case that $\mathcal{O}$ approximates an exact 1/4-BPS or 1/8-BPS operator in this regime. In this sense the $\mathcal{O}$-type operators are not 'near-BPS', but they are effectively so for any computation that depends only on the conformal dimension and R-symmetry quantum numbers. We call these operators 'nearly-BPS'.

We wish to clarify a subtlety that might be confusing when comparing the two/three angular momenta case to the BMN case. In the latter,

[12] The assumption that $J, J' \ll \sqrt{N}$ is compatible with our other assumption that $J, J' \gg \sqrt{\lambda}$, and the two together imply that the IIB string theory is weakly coupled.

[13] See the review [75] and references therein for a more complete list of work along this direction.

[14] In the free theory, $\lambda = 0$, there exist purely double-trace 1/4-BPS and purely triple-trace 1/8-BPS operators [76]. See the appendix A.

the ground state is BPS, and it is dual to a BPS operator; (most) excitations about the ground state are non-BPS and they are dual to near-BPS operators. In the former, the ground state is non-BPS but nearly-BPS; excitations about it are obviously non-BPS but, to be consistent, they should be called near-nearly-BPS. We hope that the following tables help to clarify this issue.

BMN $(J, 0, 0)$

String Side	SYM side
Ground State is 1/2 susy	$\mathcal{O} = \mathrm{Tr} X^J$ is 1/2 BPS
Excited States are near 1/2 susy	$\mathcal{O} = \mathrm{Tr}\,(XXX...D_\mu X...XX)$ is near 1/2 BPS

Frolov-Tseytlin (J_1, J_2, J_3)

String Side	SYM side
Ground State is nearly 1/4 or 1/8 susy	$\mathcal{O} = \mathrm{Tr}\left(X^{J_1} Y^{J_2} Z^{J_3}\right)$ is nearly 1/4 or 1/8 BPS
Excited States are near nearly 1/4 or 1/8 susy	$\mathcal{O} = \mathrm{Tr}\left(XX...D_\mu X...XY^{J_2} Z^{J_3}\right)$ is near nearly 1/4 or 1/8 BPS

The big difference is that the nearly BPS operators do receive quantum corrections; they do it however in a controlled way, as they become negligible as we approach the asymptotic BPS state. To actually take the limit $\lambda/J^2, \lambda/J'^2 \to 0$ we would need to go to the free theory, $\lambda = 0$. In this case, the conformal dimension of $\mathcal{O}$ does trivially saturate a BPS bound, and therefore must belong to a shortened supermultiplet. This is possible because operators that are descendants of superconformal primaries in the interacting theory can become independent BPS operators in the free-field limit [79, 80]; put in another way, short BPS states in the free theory can join into a long one as we turn on the coupling and receive radiative corrections. Note that there will be as many of these additional shortened multiplets as are required to form a long one, so the shortening provides no protection against the generation of large anomalous dimensions: the usual claim that BPS-operators have protected conformal dimensions is not true without qualification.

What happens when the condition $J, J' \ll \sqrt{N}$ is not satisfied? On the field theory side, one needs to go beyond the planar approximation.

Moreover, as we described when discussing the BMN limit (see section III.2) single-trace operators are no longer orthogonal to multi-trace operators. On the string theory side, provided $g_s \ll 1$, single-string states remain orthogonal to multi-string states. However, the description in terms of elementary strings is likely to be inadequate. This is indeed the case for states with $J' = 0$, for which the correct semiclassical description is known to be in terms of non-perturbative, rotating, spherical D3-branes, the so-called 'giant gravitons' [81]. The operators dual to these states are not single-trace operators, but (sub)determinant operators [71]; the latter *are* approximately orthogonal to each other if J is comparable to N, and only those with $J \leq N$ are independent from each other. A similar situation will presumably hold when both J and J' are non-zero. If this is the case, then the fact that the $\mathcal{O}$-type operators are only independent if $2J' + J \leq N$ (since otherwise they can be expressed as sums of products of operators of the same type) will be irrelevant to the comparison with string theory, since these operators will only provide an accurate dual description of the corresponding string theory states if $J, J' \ll \sqrt{N}$.

III.4.7 Discussion

We have discussed why quantitative tests of the AdS/CFT conjecture that go beyond kinematics are so rare: a weak-coupling computation on one side generally corresponds to an strong-coupling computation on the other side. We have seen that an exception to this state of affairs occurs in the sector of the rotating strings considered here, for two reasons [7, 8, 10, 72]. First, the energy of the corresponding *classical* string configurations happens to admit an expansion in *positive* powers of $\lambda/(J + 2J')^2$. Second, as argued by FT, partial cancellations between the vacuum energy of bosons and fermions in sigma-model quantum corrections imply that all such corrections containing non-positive powers of λ are suppressed in the limit $J + 2J' \gg 1$. These two facts allow the comparison of the string calculation to a perturbative SYM calculation in the regime in which $J + 2J' \gg 1, \sqrt{\lambda}$.

If $J' = 0$ the strings considered here reduce to the BMN strings, that is, to point-like strings orbiting the S^5 around an equator, with angular momentum J [4]. The dual BMN operators are near-BPS operators, in the sense that they are 'close to' (that is, 'a few impurities away from') an exactly 1/2-supersymmetric operator, the so-called BMN ground-state; thus, in the BMN case, the agreement tests the AdS/CFT conjecture in a near-supersymmetric sector, and this fact is presumably responsible for the partial cancellations of sigma-model quantum corrections that are essential

for the comparison to be possible.

It had not been appreciated previously that the situation is very similar for the rotating strings discussed here with $J' \neq 0$. This is implied by the results of our work, since we have shown that these strings asymptotically become 1/4- or 1/8-supersymmetric in the limit of large angular momenta. A subtle difference between the extended strings and the BMN collapsed strings case is, however, that the operators dual to the strings with $J' \neq 0$ are not near-BPS[15] but *nearly*-BPS, in the sense that there is no exactly 1/4- or 1/8-BPS operator that these operators are close to. Supersymmetry could then be responsible for

1. making sense of computing a 1-loop correction to a tachyonic σ-model vacuum, as this vacuum becomes supersymmetric and hence stable in the limits considered,

2. suppressing all these σ-model loop corrections against the classical result,

3. the replacement $\lambda \to \lambda/J^2$ in the SYM corrections to the conformal dimensions, as they must be equal to $J_1 + J_2 + J_3$ in the limit.

The picture seems to be that it has been possible to 'dig two other safety tunnels' in the AdS/CFT parameter space, centered about nearly 1/4 or 1/8 protected operators. Figure III.3 should then be replaced by figure III.6.

We would like to emphasize that tensionless strings arise in our analysis via an ultra-relativistic approximation in which the energy due to the string tension is negligible compared to the kinetic energy. In this sense, we think of the limit $\lambda/J^2 \to 0$ as a limit in which λ is kept fixed and J is sent to infinity; this is particularly natural in view of the fact that we also need $J \gg 1$. However, one can equivalently think of this limit as fixing J to be much larger than unity and then sending $\lambda \to 0$. The latter point of view is needed if one wanted to obtain the strict limit $\lambda/J^2 = 0$ without having to consider infinite energies; we then just take $\lambda = 0$, a limit in which the rotating strings become exactly supersymmetric. This may be relevant to the correspondence between tensionless strings in $AdS_5 \times S^5$ and free $\mathcal{N} = 4$ SYM theory (see, for example, [82]). Note that the free field theory has an infinite number of global symmetries, which could correspond to the gauge symmetries of massless particles of all spin in the tensionless string

[15] Except if $J' \ll J$, in which case they are BMN operators.

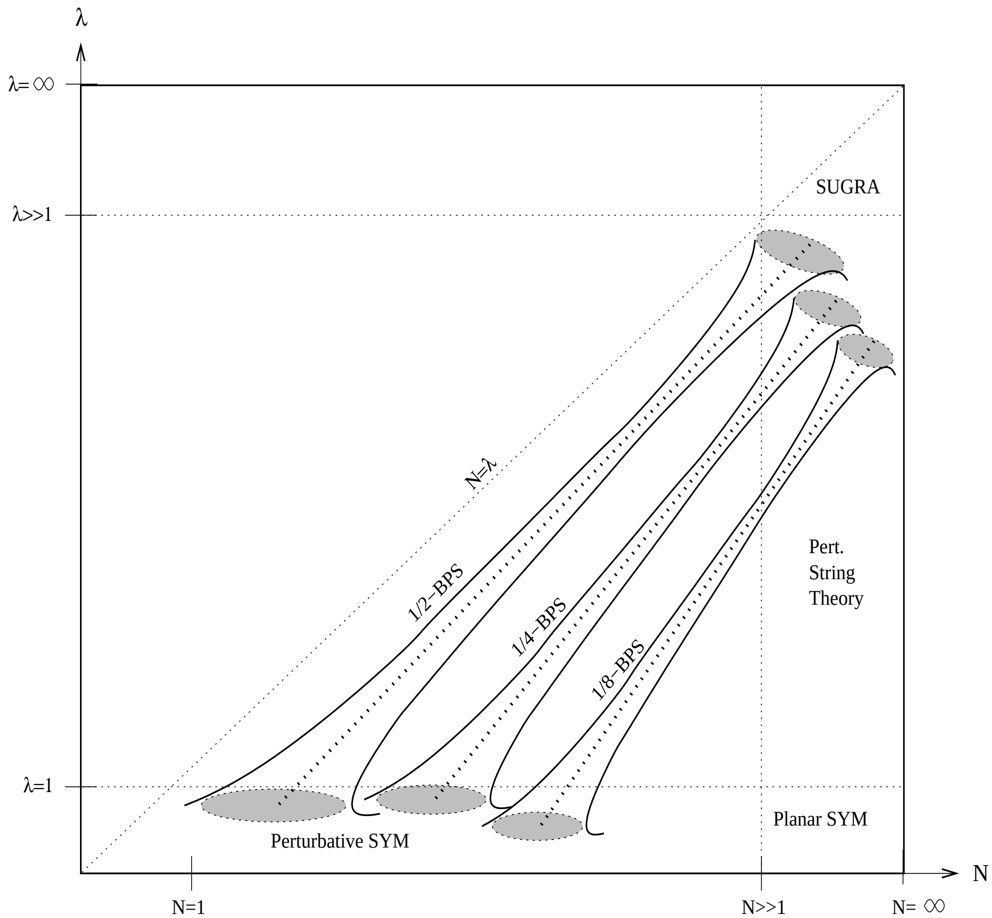


Fig. III.6: The BMN operators 'dig a safety tunnel' if we imagine placing the 1/2-BPS protected operators in its middle, and the near BPS (which receive quantum corrections) about it.

spectrum.[16]

All the configurations obtained until now in which the anomalous dimensions of strings with 2 or more angular have been successfully compared to field theory results satisfy the two properties discussed here: they become tensionless and supersymmetric in the limit in which the comparison is done. We will prove this in the appendix for all circular strings, but the same must happen for the other type of strings considered in the literature: folded strings. This is because they tend to saturate the same BPS

[16] Tensionless strings could also arise as collapsed D3-branes, for example, but such configurations could probably not be justified within a semiclassical approximation.

bounds. The only apparent exception that we are aware of is a pulsating string solution of [14], for which the classical energy matches the anomalous dimension of a SYM operator computed in perturbation theory despite the fact that this energy is not close to saturating a BPS bound. However, *it does become null* in agreement with the inevitability argument given in section III.4.5. This seems to suggest that it might be this last property, and not supersymmetry the key behind these successful comparisons. The pulsating string, however, still requires some extra work to be put into the same footing as the solutions considered here. The following questions raise suspicions about them:

1. Having an energy as far as wanted from saturating a BPS bound, would not it be reasonable to expect them to decay immediately, as there are states with the same charges but much less energy?

2. It is not clear at all that σ-model corrections will be negligible in this case, as supersymmetry is not restored asymptotically. If this happened, the classical value would be meaningless and the comparison would be just a coincidence.

The second point is a problem of special relevance after the results of [18]. In this paper, they consider the two-angular momenta strings rotating in backgrounds which are dual no non-supersymmetric field theories (so, obviously, neither the background nor the embedding of the string preserve any supersymmetry). In a first step, they find that the energy is a regular expression in the field theory coupling, so that it looks like we are in a situation were supersymmetry has nothing to do but still the comparison might be possible. However, they do compute the 1-loop σ-model correction and what they find is that, not only it is not subleading with respect to the classical contribution, but it is also not regular in the coupling! One might then suspect that it is the generic case that, in sectors completely unrelated to supersymmetry, the GKP method is not suitable for truly testing the AdS/CFT correspondence.

We would like to mention that a nice recent line of research [83] consists on trying to recover the σ-model Lagrangian directly from the field theory 1-loop operator. This would provide a comparison independent of the particular solution considered.

III.5 Stable non-BPS AdS branes

[17] In this section we conclude our analysis of stringy physics in $AdS \times S$-like backgrounds. The results that we provide are another good example of the intuition provided by the dual open/closed string pictures of D-branes.

Field theories in anti-de Sitter space have the curious feature that tachyonic modes do not lead to (perturbative) instabilities as long as they satisfy the Breitenlohner-Freedman (BF) bound [19, 20, 21]. This result has an interesting application to the field theories arising from fluctuations of branes in spacetimes of the form $AdS_{p+2} \times X_n$ for compact n-dimensional manifolds X_n. Given the existence of a 'minimal' n'-surface Σ embedded in X_n, with $n' \leq n$, there exists a minimal $AdS_{p'+2} \times \Sigma$ submanifold of $AdS_{p+2} \times X_n$, for $p' \leq p$, that is a candidate 'AdS vacuum' for a $(p' + n')$-brane.

Ambient space	Embedding map
$AdS_{p+2} \times X_n$	$AdS_{p'+2} \subset AdS_{p+2}$ $\Sigma \subset X_n$

Although Σ is a 'minimal' surface in X_n it need not have minimal volume within its homology class; it could instead have maximal volume, as would be the case for a maximal $\Sigma = S^{n'}$ in $X_n = S^n$, and in this case there exist fluctuations of Σ that *decrease* its volume. Any such fluctuation will lead to a tachyonic mode on $AdS_{p'+2}$, which would normally lead to an instability, causing the cycle Σ to contract to a lower-dimensional cycle. However, this will not happen (at least perturbatively) if the tachyonic mode satisfies the BF bound.

The canonical example of this phenomenon, which led to the AdS/dCFT correspondence [84, 85, 86], is the $AdS_4 \times S^2$ embedding of a D5-brane in the $AdS_5 \times S^5$ background of IIB superstring theory. In this case stability is guaranteed by the partial preservation of supersymmetry, and this feature is shared by all previously studied examples of stable 'AdS branes' [87, 88]. Any extension of the AdS/dCFT correspondence to non-supersymmetric defect field theories would presumably require a stable but non-supersymmetric, and hence 'non-BPS', AdS-brane. Motivated by this possibility, we examine the stability of a general class of AdS embeddings of D-branes and M-branes in background spacetimes of the form $AdS_{p+2} \times X_n$ with

[17] The remaining part of this chapter is based on unfinished work with D. Mateos and P.K. Townsend.

$X_n = S^q \times T^r$. We take $\Sigma = \Sigma_{q'} \times T^{r'}$ and choose $\Sigma_{q'}$ to be a minimal surface in S^q and $T^{r'}$ to be a minimal surface in T^r. This set-up is sufficiently general to include all previous supersymmetric AdS embeddings and, not surprisingly, the results of the perturbative analysis confirm the stability expected on the grounds of supersymmetry. Also included are many non-supersymmetric AdS embeddings, most of which are shown to violate the BF bound. However, we do find some new candidates to stable, but non-supersymmetric, AdS embeddings. We cannot be completely conclusive at this stage however, as these new cases require some extra work to determine their stability.

In all cases studied here, the $AdS_{p+2} \times X_n$ background can be viewed as the near-horizon limit of the supergravity fields produced by a large number, N, of other branes. This allows an interpretation of an AdS embedding as an effective supergravity description of a *planar* 'probe' brane in the presence of N 'background' branes. Consider, for example, the supersymmetric $(2|D3, D5)$ intersection on a 2-plane of a planar D5-brane with N coincident planar D3-branes. For large N we may replace the D3-branes by the D3 supergravity solution, and in the near horizon limit the D5-brane probe is found to be the $AdS_4 \times S^2$ embedding in $AdS_5 \times S^5$. The stability and supersymmetry of this embedding is thus inherited from the original flat space $(2|D3, D5)$ intersecting brane configuration. The stability of many other supersymmetric AdS embeddings can be understood in this way. Moreover, the fact that these AdS embeddings are typically members of families of stable and supersymmetric asymptotically-AdS embeddings can be understood from the fact that the probe is stable and supersymmetric at any distance from the background branes [88]. The *instability* of certain *non-supersymmetric* AdS embeddings can similarly be understood from the fact that, in the flat space picture, the force on the probe brane is repulsive.

Conversely, in the flat space picture, an *attractive* force will result in the formation of some bound state (which may or may not be supersymmetric) as the probe becomes coincident with the background branes. In the AdS picture, the probe brane cannot form a bound state with the background (because the background is fixed in this approximation) but there must still exist some minimum energy AdS embedding corresponding to coincidence of the probe with the background. In these cases one expects to find a stable but non-supersymmetric AdS embedding that, because of the attractive force, does not belong to a family of stable asymptotically-AdS embeddings. We will discuss the existence of several such stable AdS embeddings and we will confirm that they are not supersymmetric, even when the bound state expected on the basis of the flat space picture would be. This is

similar to the phenomenon of 'supersymmetry without supersymmetry' in which a brane configuration that is supersymmetric within the full string/M-theory fails to be supersymmetric within the supergravity approximation, a phenomenon that will be discussed on its own in sections IV.9.1 and VI.4.4.

However, not all AdS embeddings can be readily understood from a corresponding flat-space picture. One example discussed here is a non-singular $AdS_4 \times S^1 \times S^1$ embedding of a D5-brane in $AdS_5 \times S^5$ that has no flat space analog in terms of intersections of planar branes, although it can be considered as arising from an intersection of the background planar branes with a conical probe. As might be expected this AdS embedding is unstable. However, the *same* non-singular $AdS_4 \times S^1 \times S^1$ embedding of an M5-brane in $AdS_4 \times S^7$ (which has a similar interpretation as a conical M5-probe in an M2 background) turns out to be a candidate for a stable embedding. In this case, there is no obvious 'hidden' supersymmetry that would explain the stability.

We begin with a general analysis of the perturbative stability of AdS-branes, based on the BF bound. Our results are then applied to all known, and some new, AdS embeddings of branes in $AdS \times S$ backgrounds. Those cases in which we find candidates for perturbatively stable but non-supersymmetric embeddings involve either D-branes or the M-branes which are space-filling, so that their worldvolume gauge fields are actually sourced by the background potentials and cannot be set to zero. The complete analysis of these cases is still work in progress.

III.5.1 Stability of AdS-branes

Our concern here is with embeddings of branes in a background of the type $AdS_{p+2} \times X_n$ with $X_n = S^q \times T^r$. The metric is

$$ds^2 = R^2 \left[r^2 dX \cdot dX + \frac{dr^2}{r^2} \right] + d\Omega_q^2 + d\mathbf{Y} \cdot d\mathbf{Y} \,. \tag{III.69}$$

where $\mathbf{Y} = (Y^1, \ldots, Y^r)$ are cartesian coordinates for T^r and $d\Omega_q^2$ is the $SO(q+1)$-invariant metric on the unit q-sphere. The AdS_{p+2} metric is in horospherical coordinates with radial coordinate r and $(p+1)$-Minkowski coordinates $(X^0, X^1, \ldots, X^p)$. To allow for an arbitrary ratio of the AdS_{p+2} and S^q radii we have introduced the constant R as the $AdS_{(p+2)}$ radius.

We consider a $(p'+q'+r'+1)$-brane in this background with action

$$S = - \int_{AdS} \int_{\Sigma} \sqrt{- \det G_{ind}} \tag{III.70}$$

where G_{ind} is the induced metric. The brane vacuum is chosen to be an $AdS_{p'+2} \times \Sigma_{q'} \times T^{r'}$ embedding. We may take $(\rho, x^0, x^1, \ldots, x^m)$ as the worldvolume AdS coordinates and $(y^1, \ldots y^{r'})$ as the worldvolume $T^{r'}$ coordinates, and partially fix the worldvolume diffeomorphism invariance by the identifications

$$\begin{aligned} r &= \rho\,, \\ X^0 &= x^0, \qquad \ldots, \qquad X^m = x^m\,, \\ Y^1 &= y^1, \qquad \ldots, \qquad Y^{r'} = y^{r'}\,. \end{aligned} \tag{III.71}$$

Our choice of brane vacuum is an embedding in $AdS_{p+2}\times X_n$ that is partially specified by the conditions

$$X^{p'+1} = \cdots = X^p = 0, \qquad Y^{r'+1} = \cdots = Y^r = 0. \tag{III.72}$$

on the worldvolume X, Y scalar fields. The fluctuations of the Y fields cannot lead to instabilities because the $T^{r'}$ embedded in T^r is stable for topological reasons. However, we should consider fluctuations of the X fields, which we shall call the ‘AdS scalars’; they can be tachyonic as a result of mixing with modes arising from fluctuations of worldvolume gauge fields [86] but it will turn out that this is not relevant for the cases we consider.

To complete the specification of the brane vacuum we must specify the embedding of $\Sigma_{q'}$ in S^q. We must then consider the fluctuations of $\Sigma_{q'}$ in S^q, which are fluctuations of what we will call the ‘sphere scalars’. There are many possible topologies that $\Sigma_{q'}$ might have. For example, it is known that there exist minimal (maximal-area) 2-surfaces of arbitrary genus in S^3 [89], and even ones that divide S^3 into two parts of unequal volume [90], but explicit formulas are known (to us) only for S^2 and[18] T^2. More generally, for n-surfaces, there exist explicit formulas for S^n in any higher-dimension sphere, for T^n in S^{2n+1}, and for some products cases like $S^2 \times S^1$ in S^5. These are the cases we will consider here.

- $S^{q'} \in S^q$. In this case it is convenient to parametrize S^q by q angles $(\theta_1, \ldots, \theta_q)$ defined recursively:

$$d\Omega_q^2 = d\theta_q^2 + \sin^2\theta_q\, d\Omega_{q-1}^2. \tag{III.73}$$

  Let $\sigma_1, \ldots \sigma_{q'}$ be the remaining worldvolume coordinates. We may complete the gauge fixing by setting

$$\theta_1 = \sigma_1, \qquad \ldots, \qquad \theta_{q'} = \sigma_{q'}. \tag{III.74}$$

[18] Actually, an explicit formula is given in [89] for any surface of zero Euler characteristic, which would include a Klein bottle.

The vacuum configuration, for which the brane is wrapped on a maximal volume $S^{q'}$ in S^q, is then

$$\theta_{q'+1} = \cdots = \theta_q = \frac{\pi}{2}, \tag{III.75}$$

The induced metric is

$$\begin{aligned} ds^2 &= R^2\left[\rho^2 dx \cdot dx + \frac{d\rho^2}{\rho^2}\right] + d\mathbf{y}\cdot d\mathbf{y} \\ &+ \sin^2\theta_q ... sin^2\theta_{q'+1}\left[d\theta_{q'}^2 + \sin^2\theta_{q'}\left(d\theta_{q'-1}^2 + ...\right)\right] \end{aligned}$$

As we are concerned with fluctuations about the vacuum, we set

$$\theta_{q'+1} = \pi/2 + \xi_1\,, ...\,\theta_q = \pi/2 + \xi_{q-q'}\,. \tag{III.76}$$

In principle, ξ are fields on $AdS_{p'+2}\times\Sigma$ but the maximally tachyonic modes arise from fluctuations that are constant on Σ, so it will be sufficient to establish stability for these modes. We thus take ξ to be constant on Σ. In this case the action for small fluctuations is

$$S = vol(\Sigma)\int_{AdS}\sqrt{-\det g}\,[-1+\mathcal{L}]\ , \tag{III.77}$$

where g is the $AdS_{p'+2}$ metric of radius R, and

$$\mathcal{L} = -\frac{1}{2}g^{\alpha\beta}\partial_\alpha\xi\cdot\partial_\beta\xi + \frac{q'}{2}\xi\cdot\xi. \tag{III.78}$$

The product of the $q-q'$ fluctuation fields is the Euclidean product. From this result, we see that fluctuations are indeed tachyonic, with mass squared $-q'$. As the AdS space has radius R, the BF bound requires

$$q' \le \frac{(p'+1)^2}{4R^2}\,. \tag{III.79}$$

If this bound is not satisfied then the brane vacuum is unstable.

- $T^{q'}\in S^q$. We assume here that $q \ge 2q'-1$. We take the metric on S^q to be

$$d\Omega_q^2 = d\theta_q^2 + \sin^2\theta_q d\theta_{q-1}^2 + \cdots + \sin^2\theta_q\cdots\sin^2\theta_{2q'}d\Omega_{2q'-1}^2\,, \tag{III.80}$$

with $d\Omega_{2q'-1}^2$ defined recursively by the formula

$$d\Omega_{2q'-1}^2 = d\psi_{q'-1}^2 + \cos^2\psi_{q'-1}d\phi_{q'}^2 + \sin^2\psi_{q'-1}d\Omega_{2q'-3}^2. \tag{III.81}$$

Let $(\sigma_1, \ldots, \sigma_{q'})$ be the remaining worldvolume coordinates. We fix the remaining worldvolume diffeomorphisms by the identifications

$$\phi_k = \sigma_k, \qquad (k = 1, \ldots, q'), \tag{III.82}$$

so we have a $T^{q'}$ embedding into an $S^{2q'-1}$ in S^q. The volume of the embedded torus is

$$(2\pi)^{q'} \prod_{i=2q'}^{q} \sin^{q'} \theta_i \prod_{k=1}^{q'-1} \cos \psi_k \sin^k \psi_k \,. \tag{III.83}$$

This is maximal when

$$\begin{aligned} \theta_i &= \frac{\pi}{2}, \qquad (i = 2q', \ldots, q), \\ \psi_k &= \arctan\sqrt{k}, \qquad (k = 1, \ldots, q'-1)\,. \end{aligned} \tag{III.84}$$

To allow for fluctuations about this vacuum solution, we set

$$\begin{aligned} \theta_i &= \frac{\pi}{2} + \xi_i, \qquad (i = 2q', \ldots, q), \\ \psi_k &= \arctan\sqrt{k} + \sqrt{\frac{q'}{1+k}}\,\zeta_k\,, \end{aligned} \tag{III.85}$$

where the factors ensure canonical normalization of the kinetic terms for the fluctuation fields. Proceeding as before, we then find the following Lagrangian on $AdS_{p'+2}$ for these fields:

$$\mathcal{L} = -\frac{1}{2} g^{\alpha\beta} \left[\partial_\alpha \xi \cdot \partial_\beta \xi + \partial_\alpha \zeta \cdot \partial_\beta \zeta\right] + \frac{1}{2}\left[\xi \cdot \xi + 2\zeta \cdot \zeta\right]. \tag{III.86}$$

Note that the ζ fields are twice as tachyonic as the ξ fields. Their BF bound is

$$2q' \le \frac{(p'+1)^2}{4R^2}, \tag{III.87}$$

which is more restrictive than the corresponding bound for the ξ fluctuations. Thus stable AdS branes with toroidal embeddings in S^q are likely to be rarer than those with spherical embeddings.

- $S^2 \times S^1 \in S^5$. We consider this case as an illustration of the general 'product' case. The metric on S^5 can be written as

$$d\Omega_5^2 = d\psi^2 + \cos^2 \psi d\phi^2 + \sin^2 \psi \left[d\theta^2 + \sin^2 \theta d\Omega_2^2\right]\,. \tag{III.88}$$

We identify ϕ and the coordinates parametrizing the 2-sphere with the three worldvolume coordinates. This embeds $S^2 \times S^1$ in S^5. The volume of this embedded manifold is maximal when $\theta = \pi/2$ and $\psi = \arctan\sqrt{2}$, so we set

$$\theta = \pi/2 + c_1\xi, \qquad \psi = \arctan\sqrt{2} + c_2\zeta\,, \tag{III.89}$$

where the factors c_1, c_2 must be chosen again to ensure canonical normalization of the fluctuation fields. Proceeding as before, we find that the squares of the masses of these fields are

$$m_\xi^2 = -\frac{9\sqrt{3}}{2}, \quad m_\zeta^2 = -6. \tag{III.90}$$

If S^5 is itself embedded in a higher-dimensional sphere then there will be additional tachyonic modes with $m^2 = -3$, which is the value of m^2 for an S^3 embedding. The value of m^2 for a T^3 embedding is -6, so there is a fluctuation of the $S^2 \times S^1$ embedding (the ξ fluctuation) that is more tachyonic than any of the fluctuations of either the S^3 or T^3 embeddings.

This last case illustrates the pitfalls of some simple generalizations that might be suggested by the first two cases. Even though S^2 is more stable than T^2, the product $S^2 \times S^1$ is less stable than $T^2 \times S^1 \cong T^3$. Therefore, one can say that the most stable surface of a given dimension is a sphere, followed by a torus. We will now apply these results to specific string/M-theory brane configurations. In doing so we will need to use the appropriate value of the ratio R, which depends on the type of background brane, as given in the following table:

Brane Configuration	Background	R
M5	$AdS_4 \times S^7$	2
M2	$AdS_7 \times S^4$	1/2
D5	$AdS_5 \times S^5$	1
D5/D1	$AdS_3 \times S^3 \times T^4$	1

III.5.2 Applications to string/M-theory

There are a variety of AdS embeddings of string/M-theory branes in $AdS \times S$ backgrounds to which we can apply the stability results just obtained. This method has been previously used to establish the stability of some AdS

embeddings but here we shall consider a much larger class. In general we should also consider fluctuations of worldvolume gauge fields and the AdS scalars, but we will deal with this issue on a case by case basis. The cases we consider are conveniently divided into three categories.

- Supersymmetric embeddings. As explained in the introduction, these arise from standard supersymmetric intersections of planar D-branes or M-branes. Because of the WZ term in the worldvolume action, the derivatives of the AdS scalar mix with the gauge fields, and the mass squared of their fluctuations is obtained by diagonalization of an infinite matrix [86]. Typically some eigenvalues are tachyonic but they satisfy the BF bound and thus do not lead to any instability. This is expected from supersymmetry; indeed supersymmetry typically *requires* that some tachyonic modes arise in this way in order that all fluctuations form complete supermultiplets.

Brane conf.	Embedding	$(mR)^2$	Bound	Stability	Susy
(2|D3,D5)	$AdS_4 \times S^2$	-2	-9/4	stable	yes
(1|M2,M5)	$AdS_3 \times S^7$	-3/4	-1	stable	yes
(1|D3,D3)	$AdS_3 \times S^1$	-1	-1	stable	yes
(3|D3,D7)	$AdS_5 \times S^3$	-3	-4	stable	yes

- Non-supersymmetric spherical embeddings. These arise from non-supersymmetric intersections of planar branes. If the probe only overlaps the background branes, rather than intersecting them, then there will be a force on the probe (and the corresponding embedding will be only asymptotically AdS). This force may be attractive or repulsive.

  If the force is repulsive then the AdS embedding will be unstable; one would expect the q'-sphere to collapse to a point. In all cases of this type one may verify that the BF bound is violated for at least one of the sphere scalars considered in the previous section. In the $(0|D3, D3)$ case, there is a potential additional contribution to the kinetic terms of the fluctuations coming from the WZ term in the D3-brane action which could change the conclusion, but inspection shows that it contributes only to the cubic couplings. Also, there are other scalars (as discussed above), but their fluctuations do not mix with the fluctuations of sphere scalars (to quadratic order) so their presence cannot affect the conclusion either. Thus, instability is verified in all these cases.

  If the force is attractive then one expects stability. In all of the cases of this type listed in the table, *i.e.* $(3|D3, D5)$ and $(2|M2, M5)$, the

probe brane completely fills the background AdS space, so there are no AdS scalars. However, there is the additional problem that the worldvolume gauge fields cannot be set to zero because they are sourced by the background potentials. Under these circumstances, the determination of the embedding stability becomes more involved and it requires a careful examination of the resulting equations of motion. We cannot give a definite answer yet, but we hope to report on this in the near future. In the table we include the masses that fluctuations in these two cases would have if we could set the gauge fields to zero, and we leave the stability column with a question mark.

Brane conf.	Embedding	$(mR)^2$	Bound	Stability	Force
(3|D3,D5)	$AdS_5 \times S^1$	-1	-4	?	attract.
(0|D3,D3)	$AdS_2 \times S^2$	-2	-1	unstable	repuls.
(2|M2,M5)	$AdS_4 \times S^2$	-1/2	-9/4	?	attract.
(0|M2,M5)	$AdS_2 \times S^4$	-1	-1/4	unstable	repuls.
(3|D5/D1,D5)	$AdS_2 \times S^1 \times T^3$	-1	-1/4	unstable	repuls.

- Non-spherical embeddings. The main case to consider here is $T^{q'}$ for $q' > 1$. If one interprets the background AdS geometry in terms of background branes then the probe brane must be asymptotically conical, with $T^{q'}$ as the base of the cone. The vertex of this cone would be singular in flat space, but the singularity is removed by the 'back-reaction' of the background branes on the geometry, in a way that leads to a completely non-singular $AdS_{p'+2} \times T^{q'}$ embedding of the probe in $AdS_{p+2} \times X_n$. There is no obvious reason why such an embedding should be supersymmetric, so generically we should expect instability. Indeed, in all cases that we have considered, but one, the sphere scalar fluctuations violate the BF bound, and this is sufficient (by the argument given before) to prove instability. The same argument applies to the non-toroidal $S^2 \times S^1$ embedding listed in the table.

The one non-spherical case for which the BF bound is satisfied is the $AdS_4 \times T^2$ embedding of an M5-probe in the $AdS_4 \times S^7$ near-horizon geometry of the supergravity M2-brane. As this is AdS space-filling, there are no AdS scalars, and hence no tachyonic modes arising from the worldvolume 2-form gauge potential. The sphere scalars can be expanded in harmonics on T^2. The constant harmonic yields the maximally-tachyonic modes, with $(mR)^2 = -2$; this correspond to conformal coupling of scalars in AdS_4 and hence stability. If the embedding were supersymmetric there would be an *even* number of these

scalars. As the number is odd, the embedding is not supersymmetric. However, this case suffers from the same problem as the attractive cases considered above: it is not a consistent truncation to set the worldvolume gauge field to zero because it is sourced by the background potential. A more careful analysis is in progress.

Brane conf.	Embedding	$(mR)^2$	Bound	Stability	Force
(2|D3,D5)	$AdS_4 \times T^2$	-4	-4/9	unstable	repuls.
(1|M2,M5)	$AdS_3 \times T^3$	-3/2	-1	unstable	repuls.
(1|M2,M5)	$AdS_3 \times S^2 \times S^1$	$-\frac{9\sqrt{3}}{8}$	-1	unstable	repuls.
(2|M2,M5)	$AdS_4 \times T^2$	-1	-9/4	?	attract.

III.5.3 Discussion

Recall that the $AdS_5 \times S^3$ embedding of a D7-brane in $AdS_5 \times S^5$ has been used, via the AdS/dCFT correspondence, to couple N=2 quark multiplets to N=4 SYM theory. This is made possible by the fact that the D7 fills the AdS_5 space. A non-supersymmetric stable AdS embedding of this type would similarly enable us to make progress, via the AdS/dCFT correspondence, in understanding non-supersymmetric gauge theories, and this was one motivation for the work reported here. We have shown that there is indeed a candidate for an embedding of this type in which a D5-brane fills the AdS_5 and wraps a maximal S^1 in S^5. However, in the full string theory, which involves consideration of open strings connecting the probe D5 to the background D3-branes, we expect to find tachyons that imply an instability in which the D-branes dissolve into flux on the D5-brane, leading to a supersymmetric bound state. Of course, this is not seen in our approximation.

According to the AdS/CFT correspondence, M-theory on $AdS_4 \times S^7$ is equivalent a (2+1) CFT with $O(8)$ symmetry and 16 supersymmetries. According to the AdS/dCFT correspondence, AdS-embedded M5-brane probes are dual to defects in this CFT. The (possibly) stable non-supersymmetric $AdS_4 \times S^2$ embedding of an M5 in an M2-background that we have found might be used to add non-supersymmetric matter to the CFT, but this suggestion presumably suffers from the same type of problem alluded to above because in the context of M-theory we know that the M2-branes will dissolve in the M5-brane. However, we found one other example of a (possibly) stable non-supersymmetric AdS-filling embedding of an M5-brane in $AdS_4 \times S^7$ which merits further investigation in this respect. Note that *all* of the candidates to stable non-supersymmetric AdS embeddings that we

have found are AdS-space-filing.

We hope to report soon about the fate of these three configurations. What is clear at this stage is that the fact that the gauge fields do not vanish forces the static embedded brane's worldvolume to wiggle, and this could be related to the breaking of the R-symmetry of the dual theory by the addition of non-supersymmetric matter.

IV. ENGINEERING THE GAUGE/STRING DUALITY

This chapter is devoted to extending the AdS/CFT duality to more realistic field theories with less than 16 supercharges. The departure from flat embeddings of D-branes in flat space is presented as a set of logical steps that lead one to consider D-branes wrapping calibrated cycles of special holonomy manifolds. The chapter is quite self-contained and it discusses in some detail all the steps of the sequence:

- the twisting of supersymmetric gauge theories from a purely field-theoretical point of view (section IV.4),
- the mathematical background concerning special holonomy manifolds (section IV.5.1) and calibrated cycles (section IV.5.2),
- the relation between the worldvolume theories on wrapped D-branes and the field-theoretical twisting (section IV.6),
- the technical progress in trying to find the closed string duals provided by gauged supergravities (section IV.7.2).

We then illustrate the whole procedure with the explicit construction of the supergravity dual (in the IR) of an $SU(N)$ $\mathcal{N}=2$ susy field theory in 2+1 dimensions, as reported in [39]. Having obtained the dual, we study its non-perturbative moduli space as we did in [40]. During this process we will encounter a difficulty which goes under the name of 'supersymmetry without supersymmetry'. That this was a general phenomenon in the duals of non-maximally susy field theories was reported in [42] and we carefully discuss it here in section IV.9.1 and in the chapter VI (section VI.4.4).

IV.1 More general dualities involving flat D-branes

Even if the general purpose of the chapter is to find more AdS/CFT-like dualities for more realistic field theories than $\mathcal{N}=4$ in 3+1, we first need

to understand what happens if we consider Dp-branes with $p \neq 3$ and we repeat the decoupling process. Naively we would expect to obtain a duality between

SYM with 16 susys in D=p+1	$\leftrightarrow$	IIA/IIB in the near horizon limit of the (p+1)-brane SUGRA solution

There are however new problems that enter as soon as we abandon $p = 3$. As we mentioned in section II.2, the p-brane solutions have non-constant curvature for $p \neq 3$. The dilaton is not constant either, so the coupling controlling the loop corrections of string theory becomes a function of the transverse distance to the origin. Therefore the ranges of validity become more complicated here, as they are not the same for all points of the background. The limit in the supergravity side is always taken in such a way that we retain excitations in the throat that are dual to finite energy configurations of the field theory. We keep finite, for example, the mass of the W-bosons

$$M_W = \frac{r}{l_s^2} \equiv U = \text{fixed}\,. \tag{IV.1}$$

It is worth writing down in one equation the near-horizon limit of a general p-brane supergravity solution (we skip the RR forms)

$$\begin{aligned} ds^2 &= \alpha' \left(\frac{U^{(7-p)/2}}{g_{YM}\sqrt{d_p N}}\, dx_{0,p} + \frac{g_{YM}\sqrt{d_p N}}{U^{(7-p)/2}} \left[dU^2 + U^2 d\Omega_{8-p}^2 \right] \right) \\ e^{\phi} &= (2\pi)^{2-p} g_{YM}^2 \left(\frac{g_{YM}^2 d_p N}{U^{7-p}} \right)^{\frac{3-p}{4}} , \end{aligned} \tag{IV.2}$$

All quantities were defined in section II.2, but we recall here that

$$g_{YM}^2 = (2\pi)^{2-p} g_s l_s^{p-3}\,. \tag{IV.3}$$

The curvature scalar for these backgrounds can be easily computed and will be needed below

$$l_s^2\, \mathcal{R} \sim \frac{U^{\frac{3-p}{2}}}{\sqrt{g_{YM}^2 N}}\,. \tag{IV.4}$$

The supergravity solution will be valid in the regions where we can simultaneously keep $l_s^2\, \mathcal{R}, e^{\phi} \ll 1$.

How do we take the limit in the open string picture of the system? As in the D3 analysis that led to the AdS/CFT, we should

1. decouple the open and closed string massive modes, so $l_s \to 0$,
2. decouple the open/closed interactions, so $l_P = g_s^{1/4} l_s \to 0$,
3. obtain a finite g_{YM} in the low energy effective action of S_{open}.

The third condition can be used together with (IV.3) to obtain that $g_s \sim l_s^{3-p}$ in the limit, so that it diverges for $p > 3$. Using this scaling in the second condition, we obtain that open/closed string interactions decouple only if

$$l_P \sim l_s^{\frac{7-p}{4}} \to 0, \text{ as } l_s \to 0 \quad \Rightarrow \quad p < 7\,. \tag{IV.5}$$

The naive conclusion is then that it is not possible to decouple the closed strings for Dp-branes with $p \geq 7$ and that we will need dual strong coupling descriptions for D4, D5 and D6 branes. As actually the string coupling constant is U-dependent, the validity of the various descriptions will require more care as they will be energy-dependent. Having kept fixed g_{YM}, we obtain a low energy description of the system in terms of a $(p+1)$-dimensional SYM theory with 16 supercharges and gauge group $SU(N)$. The coupling constant is then dimensionful, and the dimensionless coupling that truly controls the perturbative expansion is

$$g_{eff}^2 \sim g_{YM}^2 N U^{p-3}\,. \tag{IV.6}$$

Summary:

The picture is that one has one single system. Depending on the parameters that one can freely choose, say N and g_s, *and* the energy scale U at which we probe it, the system is best described in terms of perturbative SYM, supergravity in the backgrounds (IV.2) or its strong coupling duals.

IV.2 Phase diagrams for flat D5 and D6 branes

In developing the gauge/string duality we will need to make extensive use of D5 and D6 branes in complicated target manifolds; it is therefore instructive to first understand their properties in flat space. In this section we study their phase diagrams by applying the conditions obtained above.

IV.2.1 Flat D5 Branes

The decoupling limit for N D5 branes is

$$U = \frac{r}{l_s^2} = \text{fixed}\,, \qquad g_{YM}^2 = (2\pi)^3 g_s l_s^2 = \text{fixed}\,, \qquad l_s \to 0\,. \tag{IV.7}$$

Perturbative SYM is valid when

$$g_{eff}^2 = g_{YM}^2 N U^2 \ll 1 \quad \Rightarrow \quad g_{YM} U \ll \frac{1}{\sqrt{N}}\,, \tag{IV.8}$$

which is the deep IR region. As we increase the energy we will enter in the realm of supergravity, which is valid when

$$\left.\begin{array}{lll} l_s^2 \mathcal{R} \ll 1 & \Rightarrow & g_{YM} U \gg \frac{1}{\sqrt{N}} \\ & & \\ e^{\phi} \ll 1 & \Rightarrow & g_{YM} U \ll \sqrt{N} \end{array}\right\} \Rightarrow \quad \frac{1}{\sqrt{N}} \ll g_{YM} U \ll \sqrt{N}\,. \tag{IV.9}$$

It is in that region that we can trust the supergravity solution. As we increase the energy so that $g_{YM} U \gg \sqrt{N}$, the string coupling becomes large and we need to perform an S-duality, where the N D5-branes are turned into N IIB NS5-branes. The dual background is then

$$ds_{NS5}^2 \;=\; dx_{0,5}^2 + g_s N \alpha' \left(\frac{dU^2}{U^2} + d\Omega_3^2 \right)\,, \tag{IV.10}$$

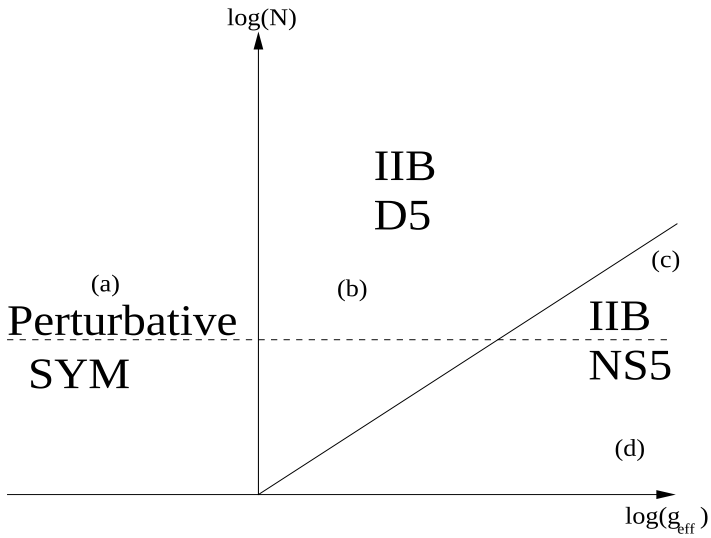
$$e^{\phi} \;=\; \left(\frac{(2\pi)^3 N}{g_{YM}^2 U^2} \right)^{1/2}\,. \tag{IV.11}$$

This is the familiar near horizon region of the full N NS5-branes solution, which would be

$$ds_{NS5,full}^2 \;=\; dx_{0,5}^2 + e^{2\phi} \left(dU^2 + U^2 d\Omega_3^2 \right)\,, \tag{IV.12}$$

$$e^{2\phi_{full}} \;=\; e^{2\phi(\infty)} + \frac{g_s N \alpha'}{U^2}\,. \tag{IV.13}$$

This system presents a problem that we had not encountered in the D-brane backgrounds before. The metric (IV.12) exhibits an infinite throat as $U \to 0$, a limit in which the radius of the S^3 remains finite and where we recover the $\mathbb{R}^{1,6} \times S^3$ geometry (IV.10). However, it is readily checked that massive geodesics can propagate in a finite proper time along the throat, which means that they do not decouple and that extra degrees of freedom should be added to a description in terms of (IV.10). The best way to deal with this case is to go back to the string σ-model in this background and realize that it is actually an exact CFT. The problem of geodesics escaping along the throat is then seen as the lack of decoupling of some stringy states in the low energy limit, where one obtains what has been named a 'little string theory'. The following diagram (extracted from [51]) summarizes the phases of the D5-branes,



(IV.14)

IV.2.2 Flat D6 branes

We now repeat a similar analysis for the N D6-branes system. The near horizon limit is

$$U = \frac{r}{l_s^2} = \text{fixed}\,, \qquad g_{YM}^2 = (2\pi)^4 g_s l_s^3 = \text{fixed}\,, \qquad l_s \to 0\,. \tag{IV.15}$$

In this limit, the radius of the M-theory circle diverges

$$R_{11} = g_s^{2/3} l_s \sim l_s^{-1} \to \infty\,, \tag{IV.16}$$

and the system should then be described by going to 11 dimensions. Once again, this the naive expectation as it depends on the energy at which we probe the system. Let us be more careful; repeating the steps above we find that perturbative SYM is valid when

$$g_{eff}^2 = g_{YM}^2 N U^3 \ll 1 \;\; \Rightarrow \;\; g_{YM} U \ll \frac{1}{(g_{YM}^2 N)^{1/3}}\,, \tag{IV.17}$$

which is the deep IR region. As we increase the energy we will enter in the realm of supergravity, which is valid when

$$\left.\begin{array}{lcl} l_s^2 \mathcal{R} \ll 1 & \Rightarrow & g_{YM} U \gg \frac{1}{(g_{YM}^2 N)^{1/3}} \\ e^{\phi} \ll 1 & \Rightarrow & g_{YM} U \ll \frac{N}{g_{YM}^{2/3}} \end{array}\right\} \Rightarrow \;\; \frac{1}{(g_{YM}^2 N)^{1/3}} \ll U \ll \frac{N}{g_{YM}^{2/3}}\,.$$

It is in that region that we can trust the IIA supergravity solution. As we increase the energy so that $g_{YM}U \gg \sqrt{N}$, the string coupling becomes large and the correct version of equation (IV.16) becomes

$$R_{11}(U) = e^{\frac{2}{3}\phi}\, l_s \gg l_s\,, \tag{IV.18}$$

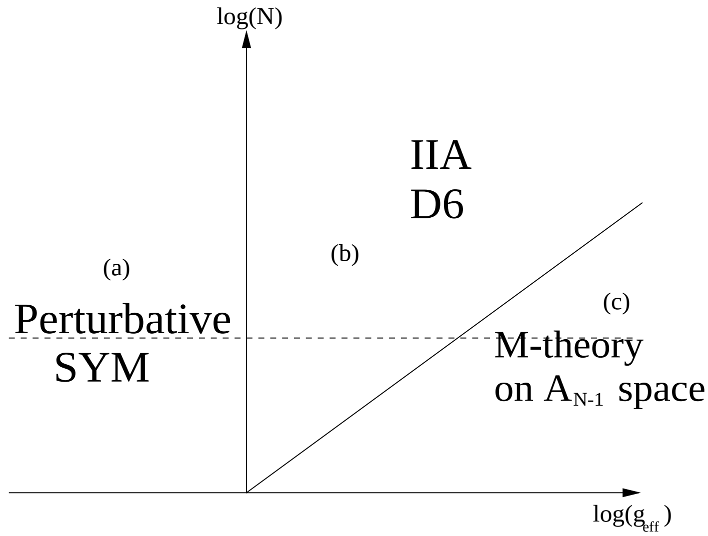
so that we have to uplift the IIA solution to M-theory. As we will thoroughly discuss in section IV.8 this leads to a solution of 11d supergravity which is purely gravitational (it does not involve any gauge field) and describes an ALE space with an $SU(N)$ singularity. Defining $y^2 = 2Ng_{YM}^2 U/(2\pi)^4$ we obtain

$$\begin{aligned} ds^2_{11d} &= dx^2_{0,6} + dy^2 + y^2\left(d\theta^2 + \sin^2\theta d\varphi^2 + \cos^2\theta d\phi^2\right) \qquad \text{(IV.19)} \\ (\varphi,\phi) &\sim \left(\varphi + \frac{2\pi}{N}, \phi + \frac{2\pi}{N}\right), \qquad 0 \le \theta < \frac{\pi}{2},\ 0 \le \varphi, \phi < 2\pi\,. \end{aligned}$$

The identification in the angles makes the S^3 metric a $U(1)$ bundle over S^2 with monopole charge N. Note that the metric (IV.19) is locally flat, so that the curvature vanishes everywhere except at the singularities. However, at very large values of y (in the far UV), the proper radius of these circles becomes very large and the 11d solution can be trusted everywhere. As the proper length of the circles is of order $y/N \sim g_{YM}N^{-1/2}U^{1/2}$, and the 11d Planck length is $l_P^{(11)} = g_s^{1/3} l_s$, we will have an everywhere flat 11d background as long as

$$R_{circles} \ll l_P^{(11)} \;\;\Rightarrow\;\; U \gg \frac{N}{g_{YM}^{2/3}}\,, \tag{IV.20}$$

which means that we should trust that 11d supergravity is a good description in the UV for any N. Thus we encounter a similar problem to the one we found for the $NS5$-branes. When a massive radial geodesic in the IIA near-horizon $D6$ background runs away from the small U region, it starts seeing the extra 11th dimension and the geometry becomes flat, so that it can easily escape to infinity. The proper description should then be the whole M-theory (and not just supergravity excitations) in the ALE space background. The phase diagram is then



(IV.21)

IV.3 Moving away from flatness

At this point we have essentially exhausted the possibility of obtaining AdS/CFT-like dualities by means of flat D-branes in flat space, and we encountered that they all involved unrealistic field theories with maximal supersymmetry. The enterprize of extending these dualities to less or non supersymmetric cases is a difficult one, but it has received a lot of attention after the original AdS/CFT appeared. The process necessarily goes through a modification of the flatness of the D-brane, of the flatness of the target space, or both simultaneously. Let us summarize the attempts performed until now[1]

- The first possibility is to replace the S^5 factor by another 5d manifold X_5. As the brane is still flat we still have an AdS factor; this means that *the field theory will still be conformal*. There are two well-known ways to modify $S^5 \to X_5$:
  - The easiest is to keep the transverse space to the brane flat except for singularities at some points. This is easily done by replacing the transverse $\mathbb{R}^6 \to \mathbb{R}^6/\Gamma$, with $\Gamma \subset SO(6)$. As the radial distance is $SO(6)$-invariant, we simply obtain a near-horizon limit

[1] This list is not exhaustive, see [91] and references therein for a more exhaustive list.

$AdS_5 \times (S^5/\Gamma)$. The number of supersymmetries correspond to the fraction of the original 16 supercharges which are left invariant by Γ. The smaller Γ is the more susy is preserved. The result is

1. $\mathcal{N} = 2$ if $\Gamma \subset SU(2)$,
2. $\mathcal{N} = 1$ if $\Gamma \subset SU(3)$,
3. $\mathcal{N} = 0$ if $\Gamma \subset SU(4)$.

– A not so straightforward procedure consists on replacing the transverse space $\mathbb{R}^6$ by some other complicated manifold X_6. If one puts the metric in the standard conical (radial×compact) form,

$$ds^2_{X_6} = dr^2 + r^2 ds^2_{X_5}\,, \tag{IV.22}$$

then any choice of an Einstein metric on X_5 leads to a 6d Calabi-Yau space which, as we will see below, does not destroy all supersymmetries. All one has to do to obtain the supergravity description is to replace $d\Omega_5^2 \to ds^2_{X_5}$ in the corresponding D3-brane solution. If we just restrict our attention to cases in which X_5 is a quotient group G/Γ, then there are only 2 possibilities

1. $X_5 = SO(6)/SO(5) = S^5$, which leads to 16 supersymmetries (+16 special conformal ones),
2. $X_5 = \frac{SU(2)\times SU(2)}{U(1)} = T^{1,1}$, which leads to 8+8 supersymmetries.

- The second possibility is to replace the AdS factor, which corresponds to having a non-flat D-brane, and leads to the *breaking of conformal invariance*. There are two completely different ways of attacking this problem.

  – One can consider small deformations of the D-brane embedding which do not change its flat asymptotics. The near-horizon is then still asymptotic to AdS_5. This means that in the UV, where the energies are much larger than those of the deformation, the theory must flow to a conformal fixed point. The dual field theory is then typically an $\mathcal{N} = 4$ SYM plus some mass terms.

  This method suffers from a very basic problem if what one would like is to obtain something similar to QCD. To that aim, one must require that the masses M of the unwanted degrees of freedom decouple from the theory before the strong-coupling phase

is reached. However, the renormalization group invariant scale is

$$\Lambda_{QCD} \sim M e^{-1/N g_{YM}^2}\,, \tag{IV.23}$$

which means that this decoupling can only be obtained if we let $M \to \infty$ together with $g_{YM}^2 N \to 0$, so that Λ_{QCD} is kept fixed. Because of $g_{YM}^2 N \to 0$, there is little hope to apply this 'deformation method' to study strongly coupled pure QCD via supergravity. Insisting with supergravity is still justified, but one must keep in mind that what is really being studied is a *QCD*-like theory with a (typically infinite) set of unwanted degrees of freedom.

- The remaining part of this chapter will deal with another possibility which completely breaks conformal invariance by abandoning the D-brane flatness property, even asymptotically. Stability will be achieved by wrapping it in minimal cycles and supersymmetry will be preserved by a beautiful mechanism called twisting. The supergravity description will then deal with spaces which have nothing to do with AdS. These models will show similar problems to the 'deformation method' ones, although some proposed way outs have been more or less successful in this respect.

From now on we will concentrate only on the last mentioned possibility.

IV.4 Twisting gauge theories

If our aim is to preserve less supersymmetry by considering curved branes in curved manifolds, the first step is to understand how to deal with a supersymmetric *field* theory in a curved space. This section is purely field theoretical and hopes to provide an understanding of what will come next, when we wrap branes in complicated spaces.

The problem of formulating susy theories in curved spaces is a difficult one. We are used to having supersymmetric theories in curved backgrounds, but these are *supergravity* theories in which the presence of spin two particles is a consequence of global supersymmetry being promoted to a local one. On the other hand we are used to formulating all kind of supersymmetric field theories (with all spins ≤ 1 and number susys ≤ 16), but they are always in flat space (or at most in spaces of constant curvature like AdS or dS). One naive way of illustrating the difficulties of changing flat space by a curved manifold is that as soon as one replaces ordinary derivatives

by covariant ones, there are new terms appearing in the transformation of the Lagrangian which are proportional to the curvature. Maybe a more clear way to say the same is that if the supersymmetry is to be realized globally (locally would require supergravity), we need to be able to deal with covariantly constant spinors, *i.e.*

$$D_\mu \epsilon = (\partial_\mu + \frac{1}{4}\,\omega_\mu)\epsilon = 0\,, \tag{IV.24}$$

where ω_μ is the spin connection on the manifold in the spinorial representation. The large majority of manifolds we can think of do not admit any non-zero solution to (IV.24), S^2 being a simple example. We should stress here that we are just working with a quantum field theory on a curved *fixed* space; the equation (IV.24) will reappear in the following sections, but the meaning will be absolutely different, as then we will be in supergravity, where ω_μ is dynamical.

So, by now we have to face the problem that we cannot formulate a supersymmetric theory in a generic curved background due to the obstruction (IV.24). The first breakthrough was due to Witten [22], who realized that one needs to modify the various irreps in which the standard fields transform. Supersymmetric theories typically involve a global R-symmetry that rotates the susy charges and, hence, the various fields in a multiplet. Witten discovered that by redefining the Lorentz group as a mixture of the usual one with the global R-symmetry it was possible to prevent all curvature terms to appear in the susy variation of the action. This was because under the new Lorentz group, all the curvature terms appeared 'hitting' fields that are now scalars, so that $[\nabla_\mu, \nabla_\nu]\phi = 0$. The change of Lorentz irreps of the fields gave the name of *twisting* to this procedure.

We will not go any deeper into Witten's point of view because it was realized some time later [92, 93] that there was an easier way to mimic the twisting. It was shown that it was all equivalent to choosing some of the conserved R-symmetry currents and coupling them to a new non-dynamical gauge connection A_μ in such a way that the modified version of the equation (IV.24)

$$D_\mu \epsilon = (\partial_\mu + \frac{1}{4}\,\omega_\mu + A_\mu)\epsilon = 0\,, \tag{IV.25}$$

admits solutions. In other words, one is making *local* a part of the originally *global* R-symmetry. Given a curved manifold $\mathcal{M}_d$ in which we want to work, the spin connection is fixed and it transforms in representations of $G \subseteq SO(d)$. If the manifold is not generic, then G can be smaller than $SO(d)$; for example, for $\mathcal{M}_6 = \mathbb{R}^4 \times S^2$, $G = SO(2)$. If we want A_μ to

cancel a part of this spin connection, we first need to set it equal (up to a sign) to ω_μ as 1-forms. Then we must hope that the spinors transformed in an irrep of the R-symmetry with the charges fine-tuned to cancel with the spin-connection. There are indeed very few ways to gauge part of the R-symmetry so that it all finally works, and each way gives rise to a different number of preserved spinors. The name *twisting* is here justified by the way the covariant derivatives effectively act on the various fields, which is modified by the presence of A_μ. The final result is a topological field theory, in the sense that the curvature drops out of all computations, as it is effectively cancelled by A_μ. We will give in section IV.6 a nice geometrical meaning to the apparently artificially introduced A_μ and to the few various ways to gauge a part of the gauge connection.

The best way to finish this section is by giving a concrete example. Let us consider a 6d gauge theory with 16 supercharges. The minimal supermultiplet consists of a vector V_μ, four scalars ϕ^i and two complex Weyl spinors of opposite chirality $\psi^\pm$. The Lorentz group is $SO(1,5)$ and the R-symmetry is $SO(4)_R \cong SU(2)_L \times SU(2)_R$. Consider now replacing $\mathbb{R}^{1,5} \to \mathbb{R}^{1,3} \times S^2$. The spin connection is only nontrivial in the tangent bundle to S^2, so it is valued in $SO(2)_{spin} \cong U(1)_{spin}$. Before doing any twist, let us recall the precise irreps of these groups in which the fields transform.[2]

	$SO(1,5) \times SO(4)$	$SO(1,3) \times U(1)_{spin} \times SU(2)_L \times SU(2)_R$
V^μ	$(6,1)$	$(4_0,1,1) \oplus (1_{+1},1,1) \oplus (1_{-1},1,1)$
ϕ^i	$(1,4)$	$(1_0,2,2)$
ψ^+	$(4,2)$	$(2_{+1},2,1) \oplus (\bar{2}_{-1},2,1)$
ψ^-	$(4',2')$	$(\bar{2}_{+1},1,2) \oplus (2_{-1},1,2)$

Now the twist begins. To cancel partially the transformation of the fields due to the S^2 curvature, we gauge a $U(1)_R$ subgroup of the $SO(4)_R$, *i.e.* we couple this $U(1)_R$ current to an external non-dynamical gauge field A_μ which we set equal to ω_μ. There are only two topologically non-equivalent embeddings of $U(1)_R \subset SU(2)_L \times SU(2)_R$ which we consider separately.

- $U(1)_R \subset SU(2)_D$, where $SU(2)_D$ is the diagonal subgroup of $SU(2)_L \times$

[2] The subscripts $\{0, \pm 1, \pm 2, ...\}$ always denote $U(1)$ charges. In this case there is only one $U(1)$ group present, so there is no room for confusion. Even when there will be more $U(1)$'s are present, it is often clear to which one they refer.

$SU(2)_R$. So we need to understand how the $SU(2)_L \times SU(2)_R$ irreps appearing in the table above decompose under this $U(1)_R$:

$SU(2)_L \times SU(2)_R$	$SU(2)_D$	$U(1)_R \subset SU(2)_D$
$(1,1)$	1	0
$(2,2)$	$1 \oplus 3$	$0 \oplus 0 \oplus +1 \oplus -1$
$(2,1)$	2	$+1 \oplus -1$
$(1,2)$	2	$+1 \oplus -1$

Finally, we need to identify the $U(1)_{spin}$ with $U(1)_R$, *i.e.* define $U(1)_{final}$ as their diagonal subgroup, and retain only the fields which are invariant under $U(1)_{final}$. The final set of charges under the relevant groups are then:

	$SO(1,3) \times U(1)_{final}$
V^μ	$4_0 \oplus 1_{+1} \oplus 1_{-1}$
ϕ^i	$1_0 \oplus 1_0 \oplus 1_{+1} \oplus 1_{-1}$
ψ^+	$2_2 \oplus 2_0 \oplus \bar{2}_0 \oplus \bar{2}_{-2}$
ψ^-	$\bar{2}_2 \oplus \bar{2}_0 \oplus 2_0 \oplus 2_{-2}$

We see that the fields which are invariant under $U(1)_{final}$ give, from a 4d point of view, one vector 4_0, two scalars $1_0 \oplus 1_0$, and two Majorana spinors $(2_0 + \bar{2}_0) \oplus (2_0 + \bar{2}_0)$. This is precisely the $\mathcal{N} = 2$ gauge supermultiplet in four dimensions, so we expect the whole theory on $\mathbb{R}^{1,3} \times S^2$ to preserve $1/2$ of the original 16 supercharges in $\mathbb{R}^{1,5}$.

This result could have been anticipated by performing the same decomposition on the supersymmetry parameters. In the original $\mathbb{R}^{1,5}$ theory, the four supercharges transform exactly in the same irrep as ψ^+ and ψ^-, which means that we can read directly from the last table that there will only remain two Majorana supersymmetry generators, *i.e.* $\mathcal{N} = 2$.

- $U(1)_R \subset SU(2)_L \subset SU(2)_L \times SU(2)_R$. We just need to repeat the same steps. The decomposition of the various $SU(2)_L \times SU(2)_R$ irreps under this choice of $U(1)_R$ are

$SU(2)_L \times SU(2)_R$	$SU(2)_L$	$U(1)_R \subset SU(2)_L$
$(1,1)$	1	0
$(2,2)$	$2 \oplus 2$	$+1 \oplus -1 \oplus +1 \oplus -1$
$(2,1)$	2	$+1 \oplus -1$
$(1,2)$	$1 \oplus 1$	$0 \oplus 0$

We now identify $U(1)_{spin}$ with this $U(1)_R$ and consider the decomposition under the diagonal $U(1)_{final}$

	$SO(1,3) \times U(1)_{final}$
V^μ	$4_0 \oplus 1_{+1} \oplus 1_{-1}$
ϕ^i	$1_{+1} \oplus 1_{-1} \oplus 1_{+1} \oplus 1_{-1}$
ψ^+	$2_2 \oplus 2_0 \oplus \bar{2}_0 \oplus \bar{2}_{-2}$
ψ^-	$\bar{2}_{+1} \oplus \bar{2}_{+1} \oplus 2_{-1} \oplus 2_{-1}$

We see that the fields which are invariant under $U(1)_{final}$ give, from a 4d point of view, one vector 4_0 and one Majorana spinor $(2_0 + \bar{2}_0)$. This is precisely the $\mathcal{N} = 1$ gauge supermultiplet in four dimensions, so we expect the whole theory on $\mathbb{R}^{1,3} \times S^2$ to preserve 1/4 of the original 16 supercharges in $\mathbb{R}^{1,5}$.

Again, recalling that the susy parameters in the $\mathbb{R}^{1,5}$ theory transform exactly in the same irrep as ψ^+ and ψ^-, we can read directly from the last table that there will only remain one Majorana spinor supersymmetry parameter, *i.e.* $\mathcal{N} = 1$.

We have learnt that the twisting mechanism allows us to put a supersymmetric field theory in $\mathbb{R}^{1,3} \times S^2$ at the price of preserving only 1/2 or 1/4 supersymmetry. At distances much larger than the radius of the S^2 (in the IR) the theory is expected to be well described by its truncation to the massless KK modes, leading to a 4d $\mathcal{N} = 1, 2$ gauge field theory.

IV.5 D-branes wrapping cycles in special holonomy manifolds

At this point we have understood from a purely field-theoretical point of view how to put a supersymmetric field theory in a curved background. This section describes one of the most amazing geometrical interpretations that D-branes in string theory have provided of a field theory phenomenon. We will see that the extra gauge connection introduced in the previous section can be interpreted as a usual spin connection on the normal bundle to the D-branes. First, however, we need to understand which type of manifolds are candidates to accept cycles where D-branes can be supersymmetrically wrapped.

IV.5.1 Special holonomy manifolds

If the worldvolume theory of a D-brane is to be supersymmetric, the first condition is that it must be put in a background that preserves at least 1 supersymmetry. Let us then look for supersymmetric solutions of the low energy supergravity theories of IIA, IIB or M-theory[3] which preserve supersymmetry. Looking for all such possible solutions is a huge task, and a complete characterization has only been achieved for maximally supersymmetric solutions[4] [5]. Luckily, we will only need a small subset of these solutions, characterized by being *purely gravitational* and of the form

$$M_D = \mathbb{R}^{1,d-1} \times X_{d'}\,, \qquad \text{with } D\text{=10,11 and d+d'=D}\,. \tag{IV.26}$$

We will use an index notation such that

$$\begin{aligned} M_D &\rightarrow M, N = 0, 1, ..., D-1\,, \\ \mathbb{R}^{1,d-1} &\rightarrow \mu, \nu = 0, 1, ..., d-1\,, \\ X_{d'} &\rightarrow i, j = 1, ..., d'\,. \end{aligned}$$

The solutions we are looking for are also purely bosonic, which means that all the fermions are set to zero. This statement is not supersymmetric invariant in general, because fermions and bosons mix under a supersymmetry transformation. So if we look for backgrounds which do not spontaneously

[3] Although we have been mainly concerned about D-branes, the discussion that follows applies to M-theory backgrounds and M-branes wrapped on their cycles as well.

[4] By 'characterization' we understand giving the explicit solution, up to coordinate transformations.

break supersymmetry we need to require, schematically,

$$\delta_{susy}\ \text{bosons}|_{sol} \sim \text{fermions}|_{sol} = 0 \tag{IV.27}$$
$$\delta_{susy}\ \text{fermions}|_{sol} \sim \text{bosons}|_{sol} = 0\,, \tag{IV.28}$$

where the subscript '$|_{sol}$' indicates that a expression must be evaluated on the solution. As fermions$|_{sol}=0$, the first equation is always satisfied. The second, however, is a non-trivial requirement. Being purely gravitational backgrounds makes it possible to treat IIA/IIB/M-theory simultaneously, as their only non-trivial equations of motion are the Einstein equations in the vacuum, and the only non-trivial variation of the fermions is that of the gravitino $\delta\Psi_M = D_M\epsilon$.[5] Therefore, supersymmetric backgrounds of IIA/IIB/M-theory are solutions of

$$\text{Supergravity e.o.m.:} \qquad R_{MN} = 0 \ \Rightarrow\ R_{ij} = 0 \tag{IV.29}$$

$$\text{Killing spinor equation:} \qquad D_M\epsilon = 0 \ \Rightarrow\ D_i\epsilon = \left(\partial_i + \frac{1}{4}w_i\right)\epsilon = 0 \tag{IV.30}$$

Together, they imply that $X_{d'}$ must be a Ricci-flat manifold with covariantly constant spinors. Having a spinor that is parallel transported along any curve must imply a restriction on the holonomy of the manifold. In particular, by considering a closed curve, we find that the spinor is unchanged, so that the holonomy group H of the manifold must admit an invariant subspace, thus it can not be as large as $SO(d')$. This is seen explicitly by taking the commutator of (IV.30), which gives the change of an object under an infinitesimal closed path,

$$0 = [D_i, D_j]\epsilon = \frac{1}{4}R_{ij\alpha\beta}\Gamma^{\alpha\beta}\epsilon\,, \tag{IV.31}$$

and recognizing $R_{ij\alpha\beta}\Gamma^{\alpha\beta}$ as the generators of the holonomy group. So the condition of supersymmetry preservation can be rephrased in terms of H by stating that $SO(d')$ must admit at least one singlet under the decomposition of its spinorial representation in irreps of $H \subset SO(d')$. Such a manifold is called a *reduced* or *special holonomy manifold*.

One important result which will be used repeatedly is that it can be shown that the connection ∇ and the curvature tensor $\mathcal{R}$ in such manifolds are restricted

$$\nabla \in End(TM) \otimes \Lambda^1 T^*M \ \rightarrow\ \nabla \in Hol(M) \otimes \Lambda^1 T^*M\,,$$
$$\mathcal{R} \in End(TM) \otimes \Lambda^2 T^*M \ \rightarrow\ \mathcal{R} \in Hol(M) \otimes \Lambda^2 T^*M\,,$$

[5] The variation of the dilatini in type IIA/IIB is proportional to the gauge fields and derivatives of the dilaton, which are all zero.

which essentially means that the comparison of tensors/spinors of tangent spaces at different points can be performed by $Hol(M)$-rotations, not with the whole most general $SO(d')$. This means that the decomposition of fields in different irreps of $SO(d')$ under $H \subset SO(d')$ provides irreps that do not mix, and can be considered as different fields on the manifold.

We will discuss the general classification of these manifolds below. Let us first note some nice properties which are common to all of them and which resemble very much the isomorphism between homology and cohomology group, in the sense that they relate global-topological with local-differential properties of a manifold.

Let us consider the spectra of three of the most important differential operators that may be defined on a manifold:

1. the Hodge-de Rham operator $(d+\delta)^2$ acting on forms,

2. the Dirac operator $i\Gamma^a D_a$ acting on spinors,

3. the Lichnerowicz operator ∇_L acting on symmetric traceless tensors.

In a general manifold, the spectra of these operators will be unrelated. However, on a manifold with reduced holonomy we can use H instead of $SO(d')$ to classify the fields over which these operators act, as the generators of H commute with the 3 operators. This is because H is obtained by parallel-transporting the fields and the 3 operators can be written in terms of covariant derivatives. The conclusion is that the spectra of these differential operators must form a representation of H at each level.

- **Example:** Let us consider a 7d manifold X_7 with G_2 holonomy to illustrate what has been said in this section. That this preserves supersymmetry is readily seen from the decomposition of the spinorial representation under the holonomy group

$$\begin{array}{ccc} SO(7) & \rightarrow & G_2 \subset SO(7) \\ 8 & \rightarrow & 1+7 \end{array} \tag{IV.32}$$

  so that there is one spinor left-invariant by H, which is the Killing spinor. In such a manifold, if we have a scalar eigenstate ϕ of the Hodge-de Rham operator, *i.e.*

$$-\nabla^i \nabla_i \phi = \lambda \phi \,, \tag{IV.33}$$

then we can immediately obtain two eigenstates of the Dirac operator out of ϕ by

$$\chi_{\pm} = \left[\phi \pm i\lambda^{-1/2}(\nabla_i\phi)\Gamma^i\right]\eta \;\;\Rightarrow\;\; i\Gamma^i\nabla_i\;\chi_{\pm} = \pm\lambda^{1/2}\,\chi_{\pm}\,, \qquad \text{(IV.34)}$$

with η a constant spinor. Similarly, if V^i is an eigenstate of the Hodge-de Rham operator

$$(d+\delta)^2 V_i = \nabla^j\nabla_j V_i + R_{ij}V^j = \lambda V_i\,, \qquad \text{(IV.35)}$$

then we can immediately obtain two eigenstates of the Dirac operator out of V^i by

$$\chi_{\pm} = \left[iV^i\Gamma_i \pm \lambda^{-1/2}(\nabla_i V_j)\Gamma^{ij}\right]\eta \;\;\Rightarrow\;\; i\Gamma^i\nabla_i\;\chi_{\pm} = \pm\lambda^{1/2}\,\chi_{\pm}\,. \qquad \text{(IV.36)}$$

One could also give the inverse transformations from eigenspinors to eigenscalars and eigenvectors.

Having seen how the spectra are related, let us focus on the zero modes. Out of a Killing spinor ϵ we can construct forms of any degree ω_n by contracting them with gamma matrices,

$$\omega_n \;=\; \frac{1}{n!}\,\bar{\epsilon}\,\Gamma_{i_1\ldots i_n}\epsilon\;\; dx^{i_1}\wedge\ldots\wedge dx^{i_n}\,. \qquad \text{(IV.37)}$$

We are not being careful with the conventions here, but once they are taken into account, it can be seen that there are a few n such that $\omega_n = 0$ identically. However, those $\omega_n \neq 0$ constructed this way are automatically zero modes of the Hodge-de Rham operator, so that they lie in a non-trivial class in cohomology. Their corresponding homology n-cycles are therefore non-trivial neither.

- Back to the G_2 example, it can be checked that the only non-zero forms that can be constructed by (IV.37) are an ω_3 and an ω_4, both related by Hodge duality.

This is as far as we will need to go with our study of reduced holonomy manifolds for the moment. The classification of the possible holonomy groups that lead to supersymmetry and their non-zero harmonic forms are given in the table below. We also indicate how many supersymmetries would be preserved if the total manifold was an M-theory background of the style $\mathbb{R}^{1,10-d'} \times X_{d'}$.

$\dim(X_{d'})$	Holonomy group	Susy	Forms	Name
4	SU(2)	16	ω_2	Calabi-Yau
6	SU(3)	8	$\omega_2, \omega_3, \omega_4$	Calabi-Yau
7	G_2	4	ω_3, ω_4	G_2-manifold
8	SU(2) × SU(2)	8	$\omega_2, \omega_4, \omega_6$	Calabi-Yau
8	Sp(2)	6	$\omega_2, \omega_4, \omega_6$	Hyper-Kähler
8	SU(4)	4	$\omega_2, \omega_4, \omega_6$	Calabi-Yau
8	Spin(7)	2	ω_4	Spin(7)-manifold
10	SU(3) × SU(2)	2	$\omega_2, \omega_3, \omega_4, \omega_6, \omega_8$	Calabi-Yau
10	SU(5)	2	$\omega_2, \omega_4, \omega_5, \omega_6, \omega_8$	Calabi-Yau

It is instructive to work out the number of preserved supersymmetries from a simple group theory argument. Let us examine the cases of 8d manifolds. The $Spin(1,10)$ structure group is broken by the background to $Spin(1,2) \times Spin(8)$. The Majorana representation splits into irreps of the latter

$$\begin{array}{ccc} Spin(1,10) & \rightarrow & Spin(1,2) \times Spin(8) \\ 32 & \rightarrow & (2, 8_+) + (2, 8_-) \end{array} \tag{IV.38}$$

The $Spin(8)$ factor is further reduced because X_8 has special holonomy.

- If X_8 is a $Spin(7)$-manifold, then we need to further decompose

$$\begin{array}{ccc} Spin(8) & \rightarrow & Spin(7) \subset Spin(8) \\ 8_+ & \rightarrow & 1 + 7 \\ 8_- & \rightarrow & 8 \end{array} \tag{IV.39}$$

  so that we only find one singlet. There are then only 2 Killing spinors that form a doublet of $Spin(1,2)$ and a singlet of $Spin(7)$. A $2+1$ physicist would call it $\mathcal{N} = 1$.

- If X_8 is a CY_4-manifold, then we need to further decompose

$$\begin{array}{ccc} Spin(8) & \rightarrow & SU(4) \subset Spin(8) \\ 8_+ & \rightarrow & 1 + 1 + 6 \\ 8_- & \rightarrow & 4 + \bar{4} \end{array} \tag{IV.40}$$

  so that we find two singlets. There are then 4 Killing spinors that form two doublets of $Spin(1,2)$. A $2+1$ physicist would call it $\mathcal{N} = 2$.

- If X_8 is a HK_4-manifold, then we need to further decompose

$$\begin{array}{ccc} Spin(8) & \rightarrow & Sp(2) \subset Spin(8) \\ & & \\ 8_+ & \rightarrow & 1+1+1+5 \\ 8_- & \rightarrow & 4+\bar{4} \end{array} \tag{IV.41}$$

so that we find three singlets. There are then 6 Killing spinors that form three doublets of $Spin(1,2)$. A $2+1$ physicist would call it $\mathcal{N}=3$.

IV.5.2 Calibrations

Now that we know in which backgrounds we should place the D-branes if we want to preserve supersymmetry, let us classify the kind of cycles that they can wrap. For that we need some mathematical background first. The plan of this section is to

1. define the notion of calibration and calibrated cycles,
2. discuss which calibrations admit the special holonomy manifolds,
3. prove the isomorphism between certain calibrated cycles and supersymmetric cycles.

IV.5.2.1 Definitions and properties of calibrations

Definition. A calibration on a Riemannian manifold $X_{d'}$ is a p-form ω satisfying

$$d\omega = 0\,, \tag{IV.42}$$
$$\omega|_{\xi^p} \leq vol|_{\xi^p}\,, \qquad \forall \xi^p\,, \tag{IV.43}$$

where ξ^p is any tangent p-plane and vol is the volume form on the cycle induced from the metric on $X_{d'}$. A p-cycle Σ_p is calibrated by ω if the inequality (IV.43) is saturated for all tangent planes to Σ_p. The main physical intuition of calibrated cycles comes from the fact that they minimize the volume within their homology class. This is easily proven by considering a calibrated cycle Σ_p and any other cycle in the same homology class Σ'_p so that $\Sigma_p - \Sigma'_p = \partial \Xi_{p+1}$. Then

$$\mathrm{Vol}(\Sigma_p) = \int_{\Sigma_p} \omega = \int_{\Xi_{p+1}} d\omega + \int_{\Sigma'_p} \omega = \int_{\Sigma'_p} \omega \leq \mathrm{Vol}(\Sigma'_p)\,. \tag{IV.44}$$

The first equality follows from Σ being calibrated. The second uses Stokes theorem. The third follows from the closure of ω and the fourth from (IV.43).

This is a rather deep result. The problem of finding minimal surfaces in a given space has been much studied in the mathematical literature and its non-simplicity arises from the fact that it requires solving a second order differential system. The problem of finding calibrated cycles is a first order one, as determining whether Σ is calibrated or not depends only on the embedding map and the tangent spaces to Σ. Calibrated geometry is a fertile source of examples of minimal submanifolds. It will come as no surprise that the equations we will have to solve when wrapping a brane on a calibrated cycle will also be of first order.

IV.5.2.2 Calibrations of special holonomy manifolds

It turns out that all the special holonomy manifolds discussed in the previous section happen to admit calibrations. The harmonic forms whose existence could be derived from the existence of Killing spinors turn out to satisfy the axioms of a calibration. Being harmonic, they are closed; the second axiom can be verified case-by-case. Let us analyze these calibrations in more detail.

- On a $Spin(7)$-manifold, there exists a harmonic 4-form which is called a Cayley calibration; its corresponding calibrated four-cycles are called Cayley cycles. It can always be written in a given orthonormal frame as

$$\begin{aligned} \omega_4 &= e^{1234}+e^{1256}+e^{1278}+e^{3456}+e^{3478}+e^{5678}+e^{1357} \\ &- e^{1368}-e^{1458}-e^{1467}-e^{2358}-e^{2367}-e^{2457}+e^{2468}, \end{aligned} \tag{IV.45}$$

  where $e^{i_1\ldots i_n}=e^{i_1}\wedge\ldots\wedge e^{i_n}$.

- On a G_2-holonomy manifold there exist calibrations of degree 3 and 4 related by Hodge duality. Cycles calibrated by the first are called associative and by the latter co-associative. In an orthonormal frame we can write

$$\omega_3 = e^{246}-e^{235}-e^{145}-e^{136}+e^{127}+e^{347}+e^{567}\,. \tag{IV.46}$$

- Calabi-Yau n-folds, where n is the complex dimension, admit two classes of calibrations. The first one is given by its Kähler 2-form J

and powers of it,

$$\omega_2 = J\,, \qquad \omega_4 = \frac{1}{2} J \wedge J\,, \qquad ...\,, \omega_{2p} = \frac{1}{p!} J^p\,. \tag{IV.47}$$

The Kähler form can always be written in an orthonormal frame as

$$J = e^{12} + e^{34} + ... + e^{(2n-1)(2n)}\,. \tag{IV.48}$$

Cycles calibrated by these forms are called holomorphic because it can be proven that their embedding in the CY manifold can be given in terms of holomorphic maps.

The other type of calibrations are given by the real part of a certain holomorphic n-form. This form is always fixed up to a phase. In an orthonormal frame we can write,

$$\omega_n \;=\; \mathrm{Re}\left[e^{i\theta}\Omega_n\right]\,, \tag{IV.49}$$

with $\theta \in S^1$ and

$$\Omega_n \;=\; (e^1 + ie^2)(e^3 + ie^4)...(e^{2n-1} + ie^{2n})\,.$$

Cycles calibrated by this form are called special Lagrangian (SLAG). For a four-dimensional Calabi-Yau, *i.e.* for a CY_2, both types of calibrations have the same degree, although they do not necessarily coincide. Indeed in this case there is a third 2-form calibration. The reason is that $SU(2) = Sp(1)$, which means that the Calabi-Yau is also a Hyper-Kähler manifold. We study this case below. The next coincidence of degree appears for a CY_4. This is because $SU(4) \subset Spin(7)$, so that a CY_4 is a particular case of $Spin(7)$. In such case, the 4-form (IV.45) can be written as

$$\omega_4 = \frac{1}{2} J \wedge J + \mathrm{Re}\left[e^{i\theta}\Omega_4\right]\,. \tag{IV.50}$$

- Hyper-Kähler manifolds with real dimension less than 10 exist only in $d' = 4$ and $d' = 8$. As mentioned above, for $d' = 4$ they coincide with a CY_2. For both $d' = 4, 8$ they admit three different Kähler 2-forms which are also calibrations. For $d' = 8$, the fauna of calibrated cycles is quite large. Apart from these 3 Kähler calibrations (and their wedge-products), and having into account the sequence $Sp(2) \subset SU(4) \subset Spin(7)$, there also exist SLAG and Cayley calibrations. A cycle can be calibrated with respect to more than one of these calibrations.

In an orthonormal frame, we can always write the 3 Kähler forms of a $Sp(2)$-manifold as

$$\begin{aligned} J^1 &= e^{12} + e^{34} + e^{56} + e^{78} \\ J^2 &= e^{14} + e^{23} + e^{58} + e^{67} \\ J^3 &= e^{13} + e^{42} + e^{57} + e^{86}\,, \end{aligned}$$

and its corresponding holomorphic 4-forms

$$\begin{aligned} \Omega^1 &= \frac{1}{2} J^3 \wedge J^3 - \frac{1}{2} J^2 \wedge J^2 + i\, J^2 \wedge J^3 \\ \Omega^2 &= \frac{1}{2} J^1 \wedge J^1 - \frac{1}{2} J^3 \wedge J^3 + i\, J^3 \wedge J^1 \\ \Omega^3 &= \frac{1}{2} J^2 \wedge J^2 - \frac{1}{2} J^1 \wedge J^1 + i\, J^1 \wedge J^2\,. \end{aligned}$$

IV.5.2.3 Calibrated cycles are supersymmetric cycles

With the technology of calibrations one can easily classify the cycles that probe D- or M-branes can wrap preserving supersymmetry. Such cycles are called *supersymmetric cycles*.

We first try to develop some intuition, and for the sake of simplicity we will start considering an M2 brane in an 11d supersymmetric background of the form discussed above $\mathbb{R}^{1,d} \times X_{d'}$. Let us single out time and write the metric as

$$ds^2_{11} = -dt^2 + g_{ij}dx^i dx^j\,, \qquad i,j = 1, ..., 10. \tag{IV.51}$$

In the absence of a background 3-form, the action for the membrane is just of a Nambu-Goto type, so that the dynamics will be such that the worldvolume tries to minimize its spacetime volume. For static configurations we can fix the gauge $t = \sigma^0$ and let x^i be independent of σ^0. Doing so we can easily compute the energy of the M2 brane

$$E = T_2 \int_{\Sigma_2} d^2\sigma\, \sqrt{\det m_{\alpha\beta}}\,, \tag{IV.52}$$

where $m_{\alpha\beta}$ is the induced metric from the background to the spatial part of the worldvolume Σ_2. Therefore static configurations minimize the spatial volume of the membrane. As supersymmetry requires that the energy of a state is minimum for its given charges, we obtain that a necessary condition for preservation of supersymmetry is that the M2 wraps a calibrated cycle.

To prove that it is sufficient we should check the κ-symmetry condition for worldvolume preservation of susy, which for an M2 reads

$$\Gamma_\kappa \epsilon = \epsilon \,, \tag{IV.53}$$

$$\Gamma_\kappa = -\frac{1}{\sqrt{\det m}} \frac{1}{2!} \epsilon^{ab} \partial_a x^i \partial_b x^j \Gamma_{0ij} \,, \tag{IV.54}$$

where ϵ is an 11d Majorana spinor and Γ_i the corresponding curved Γ-matrices, *i.e.* $\{\Gamma_i, \Gamma_j\} = 2g_{ij}$. Equation (IV.53) can be thought of as an equation for the embedding map $x(\sigma)$; its solutions preserve worldvolume supersymmetry. There is a nice way to characterize its solutions. Let us consider the positive definite quantity

$$\epsilon^\dagger \frac{(1-\Gamma_\kappa)}{2} \epsilon = \epsilon^\dagger \frac{(1-\Gamma_\kappa)}{2} \frac{(1-\Gamma_\kappa)}{2} \epsilon = || \frac{(1-\Gamma_\kappa)}{2} \epsilon ||^2 \geq 0 \,. \tag{IV.55}$$

We conclude that $\epsilon^\dagger \epsilon \geq \epsilon^\dagger \Gamma_\kappa \epsilon$, with equality if and only if (IV.53) is satisfied. Let us rewrite the inequality as

$$\sqrt{\det m} \geq \epsilon^\dagger \frac{1}{2!} \epsilon^{ab} \partial_a x^i \partial_b x^j \Gamma_{0ij} \epsilon \,. \tag{IV.56}$$

Defining $\bar{\epsilon} = \epsilon^\dagger \Gamma_0$, and a 2-form ω_2 by

$$\omega_2 = -\frac{1}{2!} \, \bar{\epsilon} \, \Gamma_{ij} \epsilon \; dx^i \wedge dx^j \,, \tag{IV.57}$$

we see that (IV.56) becomes the second condition (IV.43) for ω_2 to be a calibration. It can also be proven [94] that ω_2 is close.

The conclusion is that the cycle is supersymmetric if and only if it is calibrated by the 2-form (IV.57) constructed out of the Killing spinors. This statement can actually be generalized to all the other D- and M-branes:

> An M- or D-brane can only wrap supersymmetrically a cycle that is calibrated by a p-form which can be constructed out of the background Killing spinors.

I am not aware of any example of a calibration that cannot be constructed out of Killing spinors, but if examples exist, the discussion above shows that they would not lead to supersymmetric cycles.

IV.5.2.4 A caveat on homology and homotopy

Let us just comment on a little subtlety that can be confusing. Our intuition of why D-branes are stabilized when they wrap minimal cycles is that they

do it because there is simply no way to contract any further. So our intuition would lead us to conclude that it is homotopy what matters; we just need to find a non-contractible cycle and wrap the brane around it. However, all the discussion of the previous sections is based on homology and cohomology. Indeed, it is homology what really matters for D-branes, as dictated by their WZ couplings to the various background forms. And we know that homology and homotopy form classes of equivalence which are, in general, different (see figure IV.1).

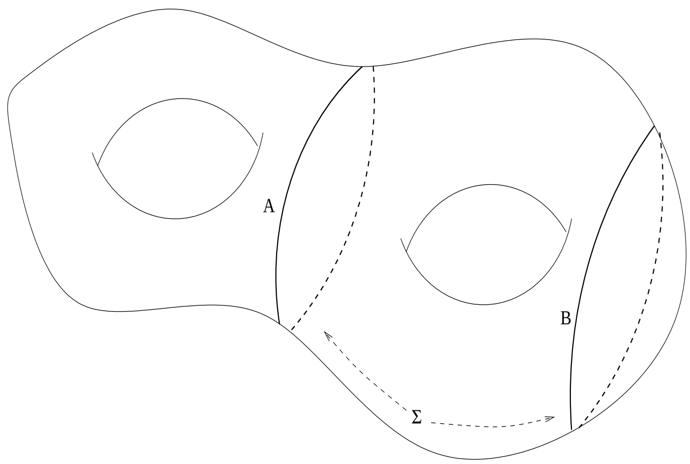


Fig. IV.1: The 1-cycles A and B are homologous as they are the boundary of the 2-surface Σ, but not homotopic as we must break A to deform it to B. In particular A and B are trivial in homology as B is a boundary, but A is not contractible.

The reason why our intuition did not fail yet is that it is true that if A_n and B_n are any two cycles of equal dimension n, then

$$A_n \text{ homotopic to } B_n \quad \Rightarrow \quad A_n \text{ homologous to } B_n\,, \tag{IV.58}$$

as can be understood by imagining A moving continuously towards B while tracing an $(n+1)$ dimensional submanifold with boundary $A \cup B$ (the theorem does not work in the inverse direction, though). In particular, we can consider B_n to be homologically trivial and then the negation of (IV.58) reads

$$A_n \text{ not trivial in homology} \quad \Rightarrow \quad A_n \text{ not trivial in homotopy}\,. \tag{IV.59}$$

So a brane that wraps a calibrated cycle (which is homologically non-trivial) is also non-contractible as our intuition required. Nevertheless, as (IV.59)

is not a both-ways implication, a brane can wrap a non-contractible cycle and be non-BPS, unstable and decay.

IV.6 The geometrical twisting

We have introduced a good amount of mathematical machinery to end up saying that branes must wrap calibrated cycles in special holonomy manifolds. As the worldvolume of D-branes carry a gauge theory, we are saying that it is possible to construct a supersymmetric gauge field theory on a curved manifold just by letting the D-brane wrap a calibrated cycle. However, we saw in section IV.4 that the inherent difficulties of formulating non-gravitational supersymmetric theories in curved spaces forced us to make use of the twisting mechanism. This involved the introduction of an external gauge field A_μ without a physical interpretation, which was needed in order to make local (a part of) the R-symmetry. The purpose of this section is to exploit the geometrical understanding that D-branes provide and show that the twisting mechanism is naturally realized by wrapping the D-branes in calibrated cycles.

Let us start with the general picture which will be illustrated below with the example of M5-branes wrapping SLAG 3-cycles. Consider a Dp-brane with worldvolume $\mathbb{R}^{1,n-1}\times\Sigma_{n'}$ embedded in IIA/IIB background of the form $\mathbb{R}^{1,d-1}\times X_{d'}$, with $X_{d'}$ a special holonomy manifold so that

Worldvolume	Target space	Embedding	
$\mathbb{R}^{1,n}\times\Sigma_{n'}$	$\mathbb{R}^{1,d-1}\times X_{d'}$	$\mathbb{R}^{1,n}\subset\mathbb{R}^{1,d-1}\,,$	$\Sigma_{n'}\subset X_{d'}$

with $p=n+n'$ and $10=d+d'$. As we know, the low energy theory of the brane worldvolume contains a set of scalars that can be interpreted as the fluctuations of the brane in the transverse space. In this case, the transverse space consists of a piece with $d-n-1$ directions along $\mathbb{R}^{1,d-1}$ and $d'-n'$ along $X_{d'}$. The scalars referring to the former must remain massless as they are the Goldstone bosons for the broken translational invariance along the flat part. The scalars referring to the latter require more work. It is useful first to decompose the tangent bundle of $X_{d'}$ as

$$\mathcal{T}_{X_{d'}}=\mathcal{T}_{\Sigma_{n'}}+\mathcal{N}_{\Sigma}\,, \tag{IV.60}$$

where $\mathcal{T}_{\Sigma_{n'}}$ is the tangent bundle to $\Sigma_{n'}$ and $\mathcal{N}_\Sigma$ its normal bundle. The latter has dimension $d'-n'$, which is the correct one to give the transverse scalars inside $X_{d'}$ the character of sections of the normal bundle.

This interpretation allows, among other things, to answer the relevant question: do we see any of these scalars in the low energy description of the D-brane? The geometric answer is simple: a scalar will remain massless, and thus will have to be included in the low energy action, if the supersymmetric cycle is not rigid. By rigid, we mean that there are no other supersymmetric cycles in its homology class which are continuously connected to it. Not being rigid means that we can move the brane away from the cycle through a set of supersymmetric cycles. Figures in IV.2 exemplify this concept.

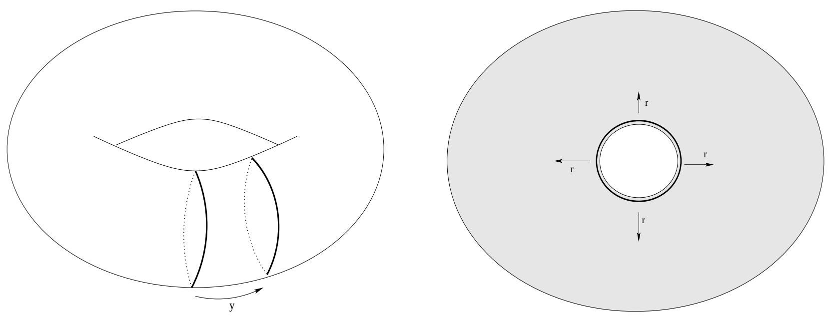


Fig. IV.2: The figure on the left shows a brane (thick line) wrapping a non-rigid cycle of a torus. Perturbations described by the scalar y are massless. On the right, a 2d cut of a manifold which has a hole in the center. If a brane wraps it, there is no other neighboring supersymmetric cycle and perturbations of the scalar r are massive.

In other words, to determine the number of scalars inside $X_{d'}$ that will be present in the effective description, we need to count the number of vector fields in $\mathcal{N}_{\Sigma_{n'}}$ that give deformations along a family of supersymmetric cycles. This can be usually translated into a purely topological argument by mapping the normal bundle to the bundle of forms of some type. Let us give two examples which may help to clarify this issue.

- Let $X_{d'} = CY_{d'/2}$ and consider a SLAG $d'/2$-cycle on it, *i.e.* a cycle parametrized by the real calibration $\omega_{d'/2}$ of (IV.49). Let us characterize the normal bundle. It is easily seen that the restriction of the Kähler form J to the cycle is zero everywhere. Therefore, for any vector field V on the cycle, we can form a 1-form $J^i j V^j$ which is normal to all vectors on Σ, and therefore belongs to $\mathcal{N}_\Sigma$. This shows that for SLAG cycles $\mathcal{T}_\Sigma \cong \mathcal{N}_\Sigma$. On the other hand, it shows that one can think of $\mathcal{N}_\Sigma$ as a frame bundle of 1-forms. A result due to Mclean [95] is that a deformation is through a set of SLAG cycles if and only if

the 1-form $J_{ij}V^j$ is harmonic. So we have a topological argument in which the first Betti number of the cycle counts the number of scalars that remain massless.

- Let $X_{d'} = CY_{d'/2}$ and consider Kähler cycles on it, which as we said are calibrated by J or powers of it. An interesting result is that, as the first Chern class of a Calabi-Yau is zero, and this number is additive under direct sum of bundles,

$$c_1[\mathcal{N}_\Sigma] = -c_1[\mathcal{T}_\Sigma]\,. \tag{IV.61}$$

  We will later deal with a particular case in which the cycle is co-dimension two (called a *divisor*). For such cases it can be shown that the normal bundle is isomorphic to a complex line bundle. It is a standard result that complex line bundles are classified by their first Chern class, so that (IV.61) completely specifies the normal bundle of divisor cycles. The notation is $\mathcal{O}(p)$ for a complex line bundle with Chern class p.

These examples show that the normal bundle to the cycle can be rather complicated depending on how the cycle is embedded in the curved manifold. We now make contact with the twisting procedure. We want to take the point of view of an observer sitting in the Dp-brane. He would interpret $X_{d'}$ not just a d'-dimensional manifold, but rather as bundle over the cycle where he is living in, the fibers being the normal vector space at each point. The connection on the normal bundle would be seen by him as an external field. Furthermore, if the observer was good at compactifying field theories, he could check that his lower dimensional SYM action can be derived from reducing 10d SYM down to his world, just as the SYM theories on Dp-branes are obtained from reduction of 10d SYM. He would then find that the surviving fields couple naturally to the normal bundle connection in a precise way dictated by how the cycle is embedded in $X_{d'}$. The action he would write down would be equivalent to the twisted actions we discussed in section IV.4. For example, the fact that for SLAG cycles we had $\mathcal{T}_\Sigma \cong \mathcal{N}_\Sigma$ is a way of rephrasing that the external gauge connection A_μ is equal to the spin connection.

We would then wonder what the possible non-equivalent twits that he can do correspond to in terms of branes wrapping cycles. The answer is that each twist corresponds to the brane wrapping different cycles in (possibly) different special holonomy manifolds. Let us reinterpret the examples we gave in section IV.4, which corresponded to 6d field theories in $\mathbb{R}^{1,3}\times S^2$. We

associate them to the field theory on a $D5$-brane in a 10d IIB background of the type $\mathbb{R}^{1,3} \times X_6$ and wrapping a cycle of X_6. The precise matching is given in the following table:

Theory	Twisting	Geometry
4d $\mathcal{N} = 2$ SYM	$U(1)_{spin} = U(1)_R$ $\subset [SU(2)_L \times SU(2)_R]_D$	D5 brane wrapped on $S^2 \subset CY_2 \times \mathbb{R}^2$
4d $\mathcal{N} = 1$ SYM	$U(1)_{spin} = U(1)_R$ $\subset SU(2)_L \subset SU(2)_L \times SU(2)_R$	D5 brane wrapped on $S^2 \subset CY_3$

The counting of preserved supersymmetries is even easier in this picture. As we saw in section IV.5.1, a CY_2, CY_3 destroys $1/2, 1/4$ of the 32 supersymmetries of Minkowsky vacuum, whereas the D5 typically breaks $1/2$ more. This makes $8, 4$ remaining supersymmetries, which from a 4d point of view is $\mathcal{N} = 2, 1$.

The counting of scalars that survive is also easier. The transverse directions to the $D5$ are all inside X_6. As the S^2 is known to be rigid in a CY, no scalars survive when $X_6 = CY_3$ and only two survive when $X_6 = CY_2 \times \mathbb{R}^2$; this is in agreement with the expected number of dimensions of their corresponding moduli spaces.

Finally, if we put N of these branes on top of each other, the gauge group of the dual theory is expected to grow to $SU(N)$ and, at distances much larger than the S^2 (in the IR) the theory becomes effectively 4d.

IV.6.1 A problem common to (almost all) supergravity solutions

In page 103 we briefly discussed the impossibility of studying less than maximally supersymmetric field theories by adding deformations to the $\mathcal{N} = 4$ Lagrangian. Essentially, supergravity was valid only in a region of the parameter space in which the added degrees of freedom do not decouple before the non-perturbative phase is reached.

Although of a qualitatively different nature, the same problem reappears in the wrapped branes arena. The intuitive understanding is clear, although one ultimately needs to check it in the final solution. The problem is that one is looking for solutions describing branes wrapping minimal cycles of the ambient space *in the limit in which these cycles are very small*, as required in order that the field theory effectively lives in the unwrapped part of the

brane. The supergravity requirement that the curvatures be small typically leads to the opposite limit in which the radii of the non-contractible spheres is large.

Again one hopes that some qualitative (and, with some luck, even quantitative) features of the wanted field theory are still captured by the supergravity approximation. Indeed, we will present some supergravity results that fit really well with the field theory expectations for some non-perturbative observables.

And finally, we remark that an exception to the rule will be provided by the closed string background dual to a noncommutative $\mathcal{N} = 1$ SYM in 3+1. We will see that the newly introduced NC scale θ can be fine-tuned in order to decouple the KK states. The reason why this scale leads to such qualitatively different results from the mass-deformations of the $\mathcal{N} = 4$ might be due to the fact that, as extensively discussed in this thesis, a NC deformation is not just a deformation through some finite set of operators but, at best, by an infinite set of them. For example, a magnetic NC deformation introduces spatial non-locality and it radically changes the classical and specially quantum properties.

Unfortunately, the limit in which θ can be used to decouple the KK modes is a limit of very large θ, so that one is left with a theory at least as unrealistic as the 16-supersymmetric one!

IV.7 How to find supergravity solutions of wrapped branes

IV.7.1 Motivation

What we have learnt in this chapter is how to deal with D/M-brane proves with curved embeddings in curved manifolds. This is the open string description that we discussed in chapter II. We can place more and more probes on top of each other in a supersymmetric way, as they all preserve the same type of Killing spinors. As the number of probes N increases the backreaction cannot be neglected and we expect that a closed string description arises, just like in the case of flat branes in flat space.

Once again, the same remark of section II.2 applies here: in the closed string description of branes in special holonomy manifolds we will not see any special holonomy manifold and, sometimes, not even branes. Maybe the simplest way to see it is that the backreaction will always involve the supergravity gauge field-strength F that couples to the brane, and this will

modify the background Killing spinor equation. Schematically

$$D_\mu \epsilon \sim (\partial_\mu + \omega_\mu + F_{\mu\alpha_1\dots}\Gamma^{\alpha_1\cdots})\,\epsilon = 0\,. \tag{IV.62}$$

So it is not ordinary covariantly constant spinors what we need, but covariant spinors in the sense of (IV.62). There has been recent progress in the geometric understanding of these spinors and in the whole supergravity solution of wrapped branes. The geometrical description is in terms G-structures, which are the proper generalization to manifolds with background fluxes of the concept of holonomy. This is however beyond the scope of this thesis.

If we were able to find the supergravity solution of one of these wrapped branes, we could try to take the near-horizon limit an expect to find an AdS/CFT-like duality relating

IR region of SYM with $\leq$16 susys in $\mathbb{R}^{1,p} \times \Sigma_{p'}$	$\leftrightarrow$	IIA/IIB in the near horizon limit of the SUGRA solution describing (p+p')-branes wrapping $\Sigma_{p'}$ inside a special holonomy manifold

We will see that these dualities are hard but possible to obtain. However, if the aim is to end up with a field theory only in the $\mathbb{R}^{1,p}$ factor, we must be able to take the limit in which $vol(\Sigma_{p'}) \to 0$ on both sides. This is what will not be possible to achieve, as we discussed in section IV.6.1, being at present the main drawback against this line of research.

IV.7.2 Using gauged supergravities to find the solutions

The purpose of finding the supergravity solutions is as hard as finding exact solutions to general relativity. Despite the fact that restricting to those solutions that preserve supersymmetry turns most of the second order differential equations of motion into first order,[6] the enterprize is still a hard one. The success in this respect during the last years is partly due to the observation of Maldacena and Núñez that one may use gauged supergravities to find them.

Let us discuss why gauge supergravities work. First of all, when we focus on the geometry of a brane wrapped on a compact cycle on a compact

[6] Not all of them, as we described in section II.3.7.1.

special holonomy manifold, it turns out that its normal bundle looks like if it was non-compact. This is because the limit in which the worldvolume theory of the N probes is the twisted SYM theory in $\mathbb{R}^{1,p} \times \Sigma_{p'}$ requires $l_s \to 0$ keeping fixed the volume of the cycle and of the special holonomy manifold. This means that the theory on the worldvolume is a large volume compactification in terms of the string length (see figure IV.3). If it had not been this way, the enterprize would have been simply impossible; for example, we still lack one single metric for a compact CY_2! So if our aim is just to describe the near horizon region, the answer must be in terms of a metric for a noncompact space.

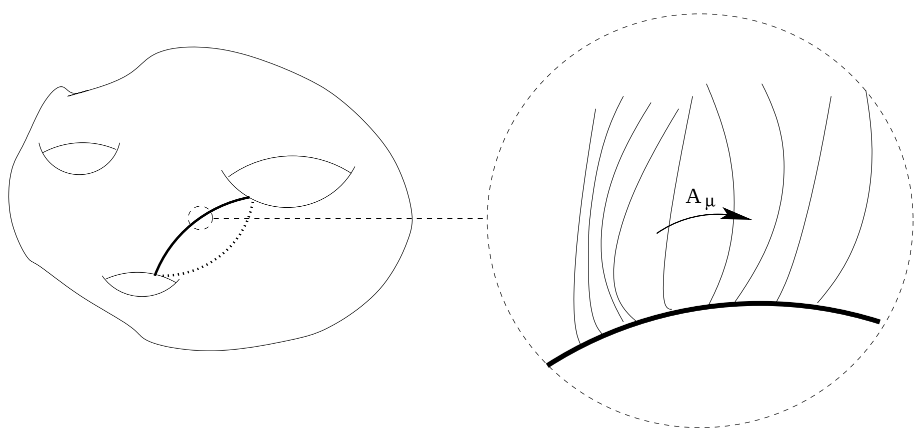


Fig. IV.3: On the left, a brane (thick line) is wrapping a nontrivial compact cycle of a complicated compact special holonomy manifold. By focusing on its worldvolume, normal bundle becomes a set of noncompact fibers; the way they curved with respect to its neighbors is controlled by the connection A_μ.

Having said this, let us guess the kind of near horizon limit that we expect. The boundary conditions on the metric must be such that it approaches $\mathbb{R}^{1,p} \times \Sigma_{p'}$ at the boundary instead of AdS. Similarly, the R-symmetry gauge fields must approach its field theory values needed in order to perform the twist. Recall that if the transverse space to the brane is n-dimensional, the R-symmetry is the isometry group the transverse S^{n-1}. It turns out that for all the cases that lead to twisted field theories [24], the supergravity fields excited by their couplings to the gauge theory belong to the multiplet which is massless upon compactification of the 10d or 11d supergravity on S^{n-1}. The conclusion is that it is much easier to give an ansatz in a supergravity theory where all the modes that are massive upon such compactification are truncated, and these are precisely the

gauged supergravities. Indeed, we need much less than a lower dimensional supergravity in which the whole $SO(n)$ is gauged, because we know that the twist requires only a part of the R-symmetry to be made local. In such a reduced supergravity, we know exactly how to perform the ansatz because the brane looks like a domain-wall there, and because the understanding of the twist from a field theoretical point of view tells us exactly which fields should be turned on. It is possibly better to explain this by finding an explicit solution, and this is the purpose of the next sections.

IV.8 Supergravity duals using D6 Branes

The purpose of this section is to find the near-horizon supergravity description of D6-branes wrapping Kähler four-cycles inside CY_3 manifolds, as reported in [39]. This should provide the AdS/CFT-like dual in the IR of a gauge theory with $\mathcal{N} = 2$ in 3d.

IV.8.1 D6 branes and M-theory

Before going to technicalities, it is worth mentioning a remarkable property that D6-branes have. The point is that their IIA supergravity solutions typically involve only the metric, the dilaton and the C_1 RR-potential, which are all fields that directly descend from the degrees of freedom of the 11d metric. So, even if their IIA supergravity solutions do not provide metrics for noncompact special holonomy manifolds, their uplift to 11d does [23]. We will repeatedly make use of this uplift, so it is good to keep the ansatz for doing so in mind:

$$\begin{aligned} ds^2_{(11)} &= e^{-\frac{2\phi}{3}} ds^2_{IIA} + e^{\frac{4\phi}{3}} \left(dx^T + C_{[1]}\right)^2 , && \text{(IV.63)} \\ A_{[3]} &= -C_{[3]} + dx^T \wedge B_{[2]}, && \text{(IV.64)} \end{aligned}$$

where A_3 is the 11d 3-form potential and C_3 the type IIA RR one.

We already used the simplest example when discussing supertubes: the uplift of the flat D6 supergravity solution in flat space (given by (II.13)-(II.15) with $p = 6$). Using (IV.63) we obtain

$$ds^2_{(11)} = dx^2_{0,6} + H\left(dr^2 + r^2[d\theta^2 + \sin^2\theta d\phi^2]\right) + R^2 H^{-1}\left(d\psi + \cos\theta d\phi\right)^2 , \tag{IV.65}$$

where N is the number of D6-branes and we recall that

$$H(r) = 1 + \frac{R}{r}, \qquad R = g_s N \sqrt{\alpha'} . \tag{IV.66}$$

This is a purely gravitational solution with metric $\mathbb{R}^{1,6} \times CY_2$, where the particular CY_2 is a Euclidean Taub-Nut space. Its near horizon limit is the ALE space discussed in section II.3.6.1.

This means that the problem of finding sugra solutions of wrapped D6 branes is doubly motivated. For a physicist, they provide non-perturbative data of SYM theories with a low degree of supersymmetry; for a mathematician they provide explicit metrics for special holonomy manifolds. The latter is a point of special relevance for special holonomy manifolds other than Calabi-Yau spaces, like G_2 and $Spin(7)$. This is because for such manifolds we do not have the analogous of Yau's theorem which states the existence and uniqueness of a Ricci-flat metric in each Kähler class. One has had to prove that several such metrics exist by brutal force until now: just going and finding them. For instance, there were only 3 examples of G_2-holonomy metrics until 2000 which were explicitly constructed in [96, 97]. After the use of wrapped D6 branes, many other explicit metrics appeared [98, 99, 100].

IV.8.2 Twisting to get $\mathcal{N}=2$ in 2+1 dimensions

Let us show that the low energy effective theory of $D6$-branes wrapping a general Kähler four-cycle inside a Calabi-Yau three-fold CY_3 is an $\mathcal{N}=2$ SYM theory in 2+1 dimensions. We just need to repeat the steps described in section IV.4.

A configuration with a D6 in flat space would have an $SO(1,6)\times SO(3)_R$ symmetry, the last group corresponding to the transverse directions to the worldvolume. The number of linearly realized supersymmetries would be 16. Consider now that our target space is instead $\mathbb{R}^{1,3}\times CY_3$, and that we wrap the $D6$ in a Kähler four-cycle inside the CY_3 in such a way that its flat directions fill an $\mathbb{R}^{1,2}\subset\mathbb{R}^{1,3}$, *i.e.*

Worldvolume	Target space	Embedding	
$\mathbb{R}^{1,2}\times\Sigma_4$	$\mathbb{R}^{1,3}\times CY_6$	$\mathbb{R}^{1,2}\subset\mathbb{R}^{1,3}$,	$\Sigma_4\subset CY_6$

The worldvolume symmetry is broken to $SO(1,2)\times SO(4)\cong SO(1,2)\times SU(2)_1\times SU(2)_2$. Being a Kähler four-cycle, its holonomy is only $U(2)$, which we identify with $SU(2)_2\times U(1)_1$, the latter being a subgroup of $SU(2)_1$.

On the other hand, the R-symmetry will be broken to a $U(1)_R\times\mathbb{R}$, with $U(1)_R$ corresponding to the two normal directions to the $D6$ that

are inside the CY_3 and $\mathbb{R}$ to the one which is in $\mathbb{R}^{1,3}$. The latter gives a massless scalar, as we can put the brane supersymmetrically anywhere in that direction. We summarize the way the various fields transform in the original and final symmetry groups in the following table. As always, we indicate the $U(1)$ charges in subscripts.

	$SO(1,6) \times SO(3)_R$	$SO(1,2) \times [SU(2)_2 \times U(1)_1] \times U(1)_R$
Scalars	$(\mathbf{1},\mathbf{3})$	$(\mathbf{1},\mathbf{1})_{(0,0)} \oplus (\mathbf{1},\mathbf{1})_{(0,1)} \oplus (\mathbf{1},\mathbf{1})_{(0,-1)}$
Spinors	$(\mathbf{8},\mathbf{2})$	$(\mathbf{2},\mathbf{1})_{(\frac{1}{2},\frac{1}{2})} \oplus (\mathbf{2},\mathbf{1})_{(-\frac{1}{2},\frac{1}{2})} \oplus (\mathbf{2},\mathbf{1})_{(\frac{1}{2},-\frac{1}{2})}$ $\oplus (\mathbf{2},\mathbf{1})_{(-\frac{1}{2},-\frac{1}{2})} \oplus (\mathbf{2},\mathbf{2})_{(0,-\frac{1}{2})} \oplus (\mathbf{2},\mathbf{2})_{(0,\frac{1}{2})}$
Vectors	$(\mathbf{7},\mathbf{1})$	$(\mathbf{3},\mathbf{1})_{(0,0)} \oplus (\mathbf{1},\mathbf{2})_{(\frac{1}{2},0)} \oplus (\mathbf{1},\mathbf{2})_{(-\frac{1}{2},0)}$

The twisting can now be understood as an identification of both $U(1)$ groups, so that only those states neutral under $U(1)_D = [U(1)_1 \times U(1)_R]$ survive. These are those irreps with opposite charge with respect to both $U(1)$'s in the table. The resulting field content consists of two Weyl fermions, one scalar and one vector, which is precisely the field content of an $\mathcal{N} = 2$ $D = 3$ SUSY theory. Later, from a supergravity point of view, we will see that these are the spinors naturally selected from the requirement that our solutions be supersymmetric.

IV.8.3 BPS equations in D=8 gauged supergravity

The aim of this section is to construct a supergravity solution describing the aforementioned $D6$-brane configurations making use of the gauged supergravities, as described in section IV.7.2. For this particular case, we need to work with eight dimensional supergravity since this is the theory that results from reducing type IIA on the S^3 transverse to the D6-branes. To be more explicit, our framework will be maximal $D = 8$ gauged supergravity, obtained in [101] by dimensional reduction of $D = 11$ on an $SU(2) \cong S^3$ manifold. We proceed to very briefly mention their results and explain our notations.

Following the usual conventions, we will use the Greek alphabet to denote curved indices and Latin ones to denote flat ones. We split the D=11

indices in (μ,α) or (a,i), the first ones in the D=8 space while the second ones in the $SU(2)\cong S^3$. The bosonic field content consists of the usual metric $g_{\mu\nu}$ and dilaton Φ, a number of forms that we will set to zero, an $SU(2)$ gauge potential A^i_μ, and five scalars parametrizing the coset $SL(3,R)/SO(3)$ through the unimodular matrix L^i_α. Finally, the fermionic content consists of a 32-components gaugino ψ_μ and a dilatino χ_i.

We will need to make use of the susy transformations for the fermions

$$\begin{aligned}\delta\psi_\rho &= D_\rho\epsilon+\frac{1}{24}e^{\Phi}F^i_{\mu\nu}\Gamma_i\left(\Gamma_\rho{}^{\mu\nu}-10\delta_\rho{}^{\mu}\Gamma^\nu\right)\epsilon-\frac{g}{288}e^{-\Phi}\epsilon_{ijk}\Gamma^{ijk}\Gamma_\rho T\epsilon \qquad \text{(IV.67)}\\ \delta\chi_i &= \frac{1}{2}\left(P_{\mu ij}+\frac{2}{3}\delta_{ij}\partial_\mu\Phi\right)\Gamma^j\Gamma^\mu\epsilon-\frac{1}{4}e^{\Phi}F_{\mu\nu i}\Gamma^{\mu\nu}\epsilon\\ &\quad -\frac{g}{8}\left(T_{ij}-\frac{1}{2}\delta_{ij}T\right)\epsilon^{jkl}\Gamma_{kl}\epsilon \qquad \text{(IV.68)}\end{aligned}$$

The definitions used in this formulae are

$$D_\mu\epsilon = \left(\partial_\mu+\frac{1}{4}w^{ab}_\mu\Gamma_{ab}+\frac{1}{4}Q_{\mu ij}\Gamma^{ij}\right)\epsilon\,, \qquad \text{(IV.69)}$$

$$P_{\mu ij}+Q_{\mu ij} \equiv L^\alpha_i\left(\delta_\alpha{}^\beta\partial_\mu-g\epsilon_{\alpha\beta\gamma}A^\gamma_\mu\right)L_{\beta j}\,, \qquad \text{(IV.70)}$$

$$T^{ij} = L^i_\alpha L^j_\beta\delta^{\alpha\beta}\,, \qquad \text{(IV.71)}$$

$$T = \delta_{ij}T^{ij}\,. \qquad \text{(IV.72)}$$

Notice that $SU(2)$ indices are raised and lowered with L^γ_i, *e.g.* $A^\gamma_\mu=L^\gamma_iA^i_\mu$. Finally we choose the usual γ-matrices representation given by

$$\Gamma^a=\gamma^a\otimes I\,,\qquad \Gamma^i=\gamma_9\otimes\sigma^i\,, \qquad \text{(IV.73)}$$

where γ^a are any representation of the $D=8$ Clifford algebra, $\gamma_9=i\gamma^0\cdots\gamma^7$, and σ^i are the usual $SU(2)$ Pauli matrices.

We now proceed to obtain the solution. Since we look for purely bosonic SUSY backgrounds, we must make sure that the susy transformation of the fermions (IV.67)(IV.68) vanishes. One of the ingredients that we put by hand is that the background Killing spinor is required to satisfy the same equation as the spinor in the twisted field theory. In other words, we impose that the first term in (IV.67) vanishes by itself, *i.e.* $D_\mu\epsilon=0$. The first immediate condition that we get is that the metric in the four cycle must necessarily be Einstein [102], so that

$$R_{ab}=\Lambda g_{ab}\qquad \Lambda=cte\,. \qquad \text{(IV.74)}$$

Inspired by the case in which the four-cycle is CP_2, we take the metric normalized in such a way that [7] $\Lambda = 6$. We then make the following domain-wall ansatz for the $D = 8$ metric

$$ds^2_{(8)} = e^{2f(r)} dx^2_{(1,2)} + e^{2h(r)} ds^2_{\Sigma_4} + dr^2 \,, \tag{IV.75}$$

where $ds^2_{\Sigma_4}$ is any Einstein metric on the Kähler 4-cycle that we want to choose.

Now, guided by the identifications between the normal bundle and the spin connection that we discussed in the last section, we complete our ansatz by switching on only one of the $SU(2)_R$ gauge fields, A^3_μ, so that we break R-symmetry to $U(1)_R$, and one of the scalars in L^i_α. This matrix can therefore be brought to [103]

$$L^i_\alpha = diag(e^\lambda, e^\lambda, e^{-2\lambda}) \,. \tag{IV.76}$$

Indeed, λ parametrizes the Coulomb branch of the gauge theory, as we discuss below. We choose vielbeins for the four-cycle such that the Kähler structure takes the form $J = e^0 \wedge e^3 + e^1 \wedge e^2$. In this basis, $D_\mu \epsilon = 0$ further implies the following identification between the R-symmetry gauge field and the four-cycle spin connection

$$A^3 = -\frac{1}{2g} w_{ab} J^{ab} \qquad \Rightarrow \qquad F^3 = dA^3 = -\frac{6}{g} \, J \,, \tag{IV.77}$$

and the following projections on the supersymmetry spinor [8]

$$\gamma^{\underline{r}} \epsilon \;=\; \epsilon \,, \tag{IV.78}$$
$$\gamma^{\underline{12}} \epsilon \;=\; \gamma^{\underline{03}} \epsilon \;=\; \Gamma^{\underline{12}} \epsilon \,. \tag{IV.79}$$

The projections that survive to these projections form a 4d vector space, which means that we are breaking 1/8 of the 32 background supersymmetries as expected. Finally, the remaining information that we can extract from our BPS equations is in the following set of coupled first-order differential equations for the functions of our ansatz $f(r)$, $h(r)$, for the dilaton $\Phi(r)$ and for the excited scalar $\lambda(r)$

$$3f' \;=\; \Phi' \;=\; \frac{g}{8} e^{-\Phi} (e^{-4\lambda} + 2e^{2\lambda}) - \frac{6}{g} e^{\Phi - 2h - 2\lambda} \,, \tag{IV.80}$$

$$h' \;=\; \frac{g}{24} e^{-\Phi} (e^{-4\lambda} + 2e^{2\lambda}) + \frac{4}{g} e^{\Phi - 2h - 2\lambda} \,, \tag{IV.81}$$

$$\lambda' \;=\; \frac{g}{6} e^{-\Phi} (e^{-4\lambda} - e^{2\lambda}) + \frac{4}{g} e^{\Phi - 2h - 2\lambda} \,. \tag{IV.82}$$

[7] See next section for a discussion about the case $\Lambda < 0$.

[8] Every time we write down a concrete index, we will underline it only if it is flat. Therefore, indices in (IV.77) are curved while those in (IV.78,IV.79) are flat. Also, $\{0, 1, 2, 3\}$ label coordinates in the four-cycle.

IV.8.4 Solutions of the BPS equations

For the case in which the scalar λ is constant, we could obtain the following exact solution of the BPS equations (IV.80,IV.81,IV.82)

$$e^{2\Phi} = \frac{9g^2}{2^{\frac{1}{3}}128}\, r^2\,, \qquad e^{2f} = C\, r^{\frac{2}{3}}\,, \qquad e^{2h} = \frac{27}{16} r^2\,, \qquad e^{6\lambda} = 2\,. \quad \text{(IV.83)}$$

There are two arbitrary integration constants. One of them is not shown explicitly, since it just amounts to a shift in the coordinate r. The other one is C, appearing in the solution for $f(r)$.

Note that if we had taken a negative value for Λ in (IV.74), the only difference would have been a change of sign in all last terms containing $1/g$. This translates into a change of sign in the solution for λ to $e^{6\lambda} = -2$. Hence, there is no supersymmetric solution for the cases $\Lambda < 0$.

One can now lift this solution to the original $D = 11$ supergravity by using the ansatz (IV.63). After performing a suitable redefinition of the radial variable, we obtain

$$ds^2_{(11)} = dx^2_{0,2} + 2dr^2 + \frac{1}{4} r^2 (d\theta^2 + sin^2\theta d\phi^2) + \frac{3}{2} r^2 ds^2_{\Sigma_4} + \frac{1}{2} r^2 \sigma^2\,, \quad \text{(IV.84)}$$

where [9]

$$\sigma = d\psi - \frac{1}{2}\cos\theta d\phi + \tilde{A}_{[1]}\,. \quad \text{(IV.85)}$$

Here we have defined $\tilde{A}_{[1]} = \frac{g}{2} A^3_{[1]}$, so that we have $d\tilde{A}_{[1]} = 3J$. The periodicities of the Euler angles are $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, whereas the periodicity of ψ depends on which particular four-cycle we choose, and we leave this issue for the particular examples.

This M-theory solution has the topology of $R^{1,2} \times CY_4$, the Calabi-Yau four-fold being a C^2/Z_n bundle over the Kähler four-cycle (again, n depends on the particular four-cycle chosen). Everything matches. As discussed in the introduction of this section, the uplift to M-theory had to provide an explicit metric for a special holonomy manifold. Looking at the table of special holonomy manifolds (page 112) we see that the only possibility that preserves four supercharges is a CY_4.

Our metric describes a cone, with $r = cte$ hypersurfaces being a $U(1)$ bundle over the base $S^2 \times \Sigma_4$. The particular fibration will depend again on the four-cycle chosen. As a good CY_4, the eight-dimensional metric is

[9] These metrics were obtained in [60] in a completely different approach. Here we follow their notation.

Kähler and Ricci-flat, thus it automatically provides vacuum solution of the D=11 equations.

Note that the metric has a conical singularity at $r = 0$, where the fiber, the S^2 and the four-cycle collapse to a point. One can now try to resolve this singularity by obtaining solutions in which at least one of the factor spaces in the base of the cone remains finite for $r \to 0$. This can be done here by dropping the assumption that the scalar λ is constant. We could find a more general solution to the BPS equations, which is best described by first changing the radial variable from the old r to R by

$$\frac{dr}{dR} = \left(\frac{gR}{4}\right)^{\frac{1}{2}} U^{-\frac{5}{12}}(R)\,, \tag{IV.86}$$

where

$$U(R) = \frac{3R^4 + 8l^2R^2 + 6l^4}{6(R^2 + l^2)^2}\,. \tag{IV.87}$$

There exists a whole family of solutions parametrized by the constant l given by

$$\begin{aligned} e^{6\lambda(R)} &= U^{-1}(R)\,, & \text{(IV.88)}\\ e^{4f(R)} &= \frac{g^2}{16}R^2U^{\frac{1}{3}}(R)\,, & \text{(IV.89)}\\ e^{2\Phi(R)} &= \left(\frac{gR}{4}\right)^3 U^{\frac{1}{2}}(R)\,, & \text{(IV.90)}\\ e^{2h(R)} &= \frac{3g}{8}RU^{\frac{1}{6}}(R)(R^2 + l^2)\,. & \text{(IV.91)} \end{aligned}$$

Repeating the lifting process to M-theory, we obtain the following 11d metric

$$\begin{aligned} ds^2_{11} &= dx^2_{(1,2)} + ds^2_{(8)}\,, & \text{(IV.92)}\\ ds^2_{(8)} &= U^{-1}(R)dR^2 + \frac{1}{4}R^2(d\theta^2 + sin^2\theta d\phi^2) + \frac{3}{2}(R^2 + l^2)ds^2_{\Sigma_4} \\ &\quad + U(R)R^2\sigma^2\,. & \text{(IV.93)} \end{aligned}$$

Note that for $l = 0$ this collapses to the original singular solution (IV.84). On the other hand, for $l \neq 0$ the four-cycle has blown-up, and its size remains finite at $R \to 0$, although the S^2 and the $U(1)$ fiber still collapse. Nevertheless, recall [104] that the condition for local regularity in this limit implies that *at most one* of the factors in the base of the $U(1)$ fiber can collapse. Our manifold is therefore locally regular. Globally, it will depend on the four-cycle chosen, as the following examples show.

- **Example I:** Consider the choice of a $\Sigma_4 = CP_2$. This a Kähler holomorphic cycle of codimension two in a Calabi-Yau manifold, *i.e.* a divisor. We described in section IV.6 how the normal bundle is for such cases and found that it must form a holomorphic line bundle with opposite Chern class with respect to the cycle. Given that in our conventions $c_1[CP_2] = 3$ the normal bundle must be an $O(-3)$ bundle.

  After this discussion about the global structure, we aim to make explicit all the functions that were left unspecified for being cycle-dependent. We provide the CP_2 base with the standard Fubini-Study unit metric, *i.e.*

$$ds^2_{CP_2} = \frac{1}{(1+\rho^2)^2}d\rho^2 + \frac{\rho^2}{(1+\rho^2)^2}{\sigma_3}^2 + \frac{\rho^2}{1+\rho^2}{\sigma_1}^2 + \frac{\rho^2}{1+\rho^2}{\sigma_2}^2 \,, \quad \text{(IV.94)}$$

  where σ_i are the $SU(2)$ left-invariant one forms normalized such that $d\sigma_i = \epsilon_{ijk}\sigma_j\sigma_k$. This metric is Einstein, with $R_{ab} = 6\, g_{ab}$ as required by our conventions. When we plug this metric in our M-theory solution (IV.93), we obtain that $\tilde{A}_{[1]} = -\frac{3}{2}\rho e_3$. We substitute this in (IV.85) and, applying the arguments in [104], we see that the maximum range of the $U(1)$ fiber angle must be restricted to $(\Delta\psi)_{max} = \pi$ instead of the normal 2π. We have a CP_2 bolt at the origin. This is why the $U(1)$ fibers over S^2 do not describe an S^3 (viewed as a Hopf fibration), but an S^3/Z_2.

- **Example II:** We give now an example in which the four-cycle is taken an $S^2 \times S^2$. As the metric had to be Einstein both spheres need to have the same radius. Finally, in order to normalize them such that $R_{ab} = 6\, g_{ab}$, their radii must be $r^2 = 1/6$, so that

$${ds^2}_{S^2\times S^2} = \frac{1}{6}({d\theta_1}^2 + sin^2\theta_1 {d\phi_1}^2) + \frac{1}{6}({d\theta_2}^2 + sin^2\theta_2 {d\phi_2}^2)\,. \quad \text{(IV.95)}$$

  Now $\tilde{A}_{[1]} = \frac{1}{2}\left[\cos\theta_1 d\phi_1 + cos\theta_2 d\phi_2\right]$ so, unlike before, this allows a standard range $(\Delta\psi)_{max} = 2\pi$. Hence, topologically, the manifold is a regular C^2 bundle over $S^2 \times S^2$.

IV.9 Non-perturbative physics of $\mathcal{N}=2$ in 2+1 from its supergravity dual

There is a good amount of non-perturbative qualitative (and sometimes quantitative) physics of gauge theories that can be extracted from the string duals. In the next chapter we will analyze issues like confinement or chiral symmetry breaking in both ordinary and NC $\mathcal{N}=1$ SYM in four dimensions. Here we will devote our attention to the above obtained supergravity dual of its cousin $\mathcal{N}=2$ in 3d, which corresponds to its reduction on an S^1. We will just explore the data that this dual provides about its moduli space. As discussed in sections II.1.2 and II.1.3, this can be studied by introducing a probe brane in the background created by the others and computing its effective action. As our solution has been found in 11d, the first task is to reduce it back to type IIA and then put a probe. However, in the paper where the solution (IV.92)-(IV.93) was originally constructed [39] we did not have into account that a problem usually referred to as *supersymmetry without supersymmetry* was going to be relevant in our case. Let us summarize what this problem consists on. We will then show how it affected our reduction and the incorrect conclusions that were originally derived. We then perform the correct reduction and discuss the non-perturbative moduli space of the $\mathcal{N}=2$ SYM in 2+1.

IV.9.1 Supersymmetry without supersymmetry

The fact that the radius of the eleventh dimensions is proportional to the IIA string coupling constant, $R_{11}=g_s^{2/3}l_s$, has deep consequences on the physics felt by observers in 10 or 11 dimensions. It is well known that the compactification from 11d to 10d is a consistent one for any 11d vacuum that can be viewed as a $U(1)$ bundle over some base manifold $\mathcal{M}$. The point is that M-theory states, even those of its massless supergravity sector, which are charged under $U(1)$ rotations will look massive from a IIA point of view, with masses

$$m_{IIA}\sim\frac{1}{g_s^{1/3}R_{11}}\sim\frac{1}{g_sl_s}\,, \tag{IV.96}$$

where the extra factor $g_s^{1/3}$ appears when we measure distances in terms of the string metric in 10d. An example of such states are excitations of the 11d metric which are not $U(1)$ invariant; they become IIA states which can be identified with D0-branes. These are invisible if we just do perturbative string theory. However, from an 11d point of view there is no such distinction between perturbative and non-perturbative states, which

means that a 10d observer may feel like his supersymmetry multiplets are shorter as a consequence of describing the world perturbatively.

This phenomenon was named 'supersymmetry without supersymmetry' in [105] as supersymmetry is actually present but in a nonperturbative way for observers in the compactified theory. They provided a very nice example in which 11d supergravity was reduced on $AdS_4 \times S^7$ following two routes.

- When compactified directly from 11d to 4d, the massless sector falls into $SO(8)$ $\mathcal{N}=8$ supermultiplets, and it is described by the corresponding $\mathcal{N}=8$ supergravity in 4d.

- As S^7 can be seen as a $U(1)$ bundle of CP^3, one can first go to IIA and then reduce on CP^3. The point is that in the first step, some states will disappear due to their nonperturbative condition, the consequence being that the final 4d result will appreciate less than $\mathcal{N}=8$ supercharges. Depending on the orientation of the S^7 it was shown that the 4d observer would measure either $\mathcal{N}=6$ or $\mathcal{N}=0$.

A more subtle point arises when one reduces from 11d to 10d on a bosonic vacuum that is completely $U(1)$ invariant. Of course, in the quantum theory one needs to consider the whole tower of 11d excitations, some of which will correspond to D0-branes, but here we want to concentrate on the vacuum only. Being a bosonic background, it reduces down to another bosonic background; being $U(1)$ invariant means that all its bosonic fields fit into the reduction ansatz (IV.63) and no 'non-perturbative' states are generated. But apart from the 11d supermultiplet, the reduction ansatz involves the background Killing spinors as well. The danger is that it can happen that only the Killing spinors fail to be $U(1)$ invariant. What are the consequences then? As all bosonic fields do fit in the ansatz, the 10d configuration will perfectly solve the IIA equations of motion. However, the configuration will be more or less supersymmetric depending on how many 11d Killing spinors are singlets under the $U(1)$. This problem is difficult to avoid, the safest way possibly being the brute force explicit computation of the 11 Killing spinors.

Needless to say, this problem extends to any reduction of supersymmetric bosonic solutions of any theory. The next subsections focus on the particular 11d-to-10d case, and we will see many new examples in chapter VI arising in more general $11d$ compactifications.

IV.9.2 A non-supersymmetric compactification and a zero-dimensional moduli space

We now go back to our study of the $\mathcal{N} = 2$ SYM in 2+1 via the supergravity solution. In order to analyze its moduli space, we first need to reduce the solution (IV.92)-(IV.93) from 11d to IIA, and we will do it here along the simplest possible S^1. Since the metric (IV.93) has a $U(1)$ isometry, with Killing vector ∂_ψ, we choose that direction as the M-theory circle. Using the KK ansatz (IV.63) we obtain a bosonic type IIA solution with the following values for the metric, the dilaton and the RR one-form

$$ds^2_{IIA} = e^{2\Phi/3}\left[dx^2_{1,2} + U^{-1}dr^2 + \frac{r^2}{4}(d\theta^2 + sin^2\theta d\phi^2) + \frac{3}{2}(r^2 + l^2)ds^2_{\Sigma_4}\right], \tag{IV.97}$$

$$e^{4\Phi/3} = U(r)r^2\,, \tag{IV.98}$$

$$C_{[1]} = A_{[1]} - \frac{1}{2}\cos\theta d\phi\,. \tag{IV.99}$$

Notice that the dilaton vanishes at $r \to 0$ and diverges at infinity, which means that one expects a good description with classical string theory only for small values of r. Essentially, this problem comes from the fact that our $U(1)$ fiber radius in the eleven-dimensional metric already diverged. Obtaining solutions with a finite circle at infinity would probably require an analysis beyond gauged supergravity. A different approach, based on imposing directly the required symmetries in the whole D=11 supergravity, enabled the authors of [98] to construct such kind of solutions.

Our metric is clearly singular at $r \to 0$. In order to apply the criteria for good/bad singularities of [106], one needs to put the metric (IV.97) in the Einstein frame, which just amounts to multiplying by $e^{-\frac{\Phi}{2}}$. It can be seen that g_{00} decreases (and it is bounded) as we approach the singularity, which means that excitations of fixed proper energy are seen with lower and lower energy from an observer at infinity as we approach the origin. Thus we conclude that it is a *good* one, properly describing the IR behavior of the dual theory.

Let us now put a probe brane in the background of the wrapped $D6$ that we have obtained. We consider a probe wrapping the same cycle but at some distance in the $\mathbb{R}^{1,3}$ factor, so that one can think of it as pulling one of the $D6$ apart from the others. The effective action for such a probe

in the case that $\Sigma_4 = CP_2$ is, from (II.1)-(II.3),

$$\begin{aligned} S &= -\mu_6 \int_{R^{1,2}\times CP_2} d^7\xi \; e^{-\Phi}\sqrt{-det[G + B_{[2]} + 2\pi\alpha' F_{[2]}]} \\ &\quad +\mu_6 \int_{R^{1,2}\times CP_2} [exp(2\pi\alpha' F + B) \wedge \oplus_n C_{[n]}] \end{aligned} \tag{IV.100}$$

In our solution (IV.97)(IV.99) we have $B_{[2]} = 0$ and only $C_{[1]} \neq 0$. In order to pull back our fields we choose a static gauge, in which we identify the worldvolume coordinates $\{\xi^i,\, i = 0, ..., 6\}$ with the space time coordinates $\{x^0, x^1, x^2, \rho, \tilde{\theta}, \tilde{\phi}, \tilde{\psi}\}$, the first three parametrizing $R^{1,2}$, and the other four the CP_2. We will look for the vacuum configuration and so we will set to constant the three space time coordinates normal to the brane $\{r, \theta, \phi\}$. With these choices, our formula (IV.100) becomes

$$S = -\mu_6 \; Vol\left[R^{1,2}\right] \int_{CP_2} d\rho d\tilde{\theta} d\tilde{\phi} d\tilde{\psi} \; \frac{a^{3/2}\rho^3(a + b\rho^2)^{1/2}\sin\tilde{\theta}}{8(1+\rho^2)^3}\,, \tag{IV.101}$$

where a and b are the following functions of r

$$a(r) = \frac{3}{2} r U(r)^{\frac{1}{2}}(r^2 + l^2)\,, \qquad b(r) = \frac{9}{4} r^3 U(r)^{\frac{3}{2}}\,. \tag{IV.102}$$

Looking at the integrand, which is always positive, we already see that its minimum is at $r = 0$ where, indeed, $S = 0$.

The dimension of the moduli space can be determined by looking at the kinetic terms arising from the DBI action when one allows for the transverse coordinates $\{r, \theta, \phi\}$ to depend on the flat worldvolume ones $\{\xi^0, \xi^1, \xi^2\}$. The exact expression one obtains is identical to that in (IV.101) but replacing

$$Vol\left[R^{1,2}\right] \;\rightarrow\; \int d\xi_1 d\xi_2 d\xi_3 \sqrt{det\left(\delta_{ij} + \partial_i r\partial_j r + \frac{1}{4}\partial_i\theta\partial_j\theta + \frac{1}{4}sin^2\theta\partial_i\phi\partial_j\phi\right)} \tag{IV.103}$$

Here $\{\partial_i = \frac{\partial}{\partial\xi^i},\, i = 0, 1, 2\}$. Clearly, evaluating this at the minimum $r = 0$ still makes the whole expression vanish. Hence, In this approximation we find that the moduli space is zero-dimensional!

What went wrong?

We will discuss in detail below what is known about the moduli space of the $\mathcal{N} = 2$ SYM theory and what our supergravity analysis was expected to give. Even without that knowledge, there is simple way to see that

something is going wrong and to explain why we got a zero-dimensional moduli space. Despite the fact that we reduced along a $U(1)$ isometry, *i.e.* all our bosonic fields are $U(1)$ invariant, the Killing spinors of the 11d solution are all charged under this $U(1)$. Their explicit computation will be performed in section VI.4.3 and their relevant expression is given in formula VI.101. For our purposes here we just need to note that they all depend on ψ, which means that they all acquire masses in 10d.[10] So, as explained in the previous subsection, we should expect this 10d solution to solve the IIA equations of motion but to preserve no supersymmetry at all. This is indeed the case as we explicitly checked. To conclude this caveat, the lack of supersymmetry in the background prevents one of the branes to be pulled away from the others in a supersymmetric way; in field-theoretical terms, the moduli space must be zero-dimensional. The conclusion is simply that *this IIA background is not dual to the 2+1 $\mathcal{N}=2$ SYM gauge theory.*

IV.9.3 A supersymmetric compactification and an all-loops perturbative moduli space

IV.9.3.1 The IIA solution

As the reduction along the ψ-circle destroyed all supersymmetries, it is logical to wonder whether we can find any other $U(1)$ isometry such that the Killing spinors are invariant. Again we need to advance results that will be properly obtained and discussed in section VI.4.4. It turns out that the correct reduction must be performed along the isometry generated by the Killing vector ∂_ϕ of the 11d metric (IV.92)-(IV.93). Note that both ψ and ϕ are angles on the S^3 transverse to the 4-cycle inside the CY_4. As this S^3 is squashed, it makes physical difference to distinguish between both angles. For the sake of simplicity we will restrict here to the case when the 4-cycle is an $S^2 \times S^2$ with equal radii, so that as in (IV.95),

$$ds^2_{\Sigma_4} = \frac{1}{6}\left(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2\right) . \tag{IV.104}$$

[10] The reader might complain that not having checked the 11d supersymmetry variations yet we should not claim that the 11d solution (IV.92)-(IV.93) is supersymmetric. However, this is not necessary as 11d supersymmetry is guaranteed by the fact that the solution is the uplift of a supersymmetric 8d one.

Reducing along ∂_ϕ we obtain

$$
\begin{aligned}
ds^2_{IIA} &= \left(\frac{4g_sN}{r^2 f}\right)^{-\frac{1}{2}} dx^2_{0,2} + (g_sN)^{\frac{1}{2}} \left\{ \frac{3}{2}(r^2+l^2)ds^2_{S^2\times S^2} \right. \\
&\qquad \left. + U^{-1}dr^2 + \frac{r^2}{4}[d\theta^2 + mB^2_{[1]}] \right\} , && \text{(IV.105)} \\
e^{4\Phi/3} &= \frac{g_s^{\frac{1}{3}} r^2 f}{4N} , && \text{(IV.106)} \\
C_{[1]} &= -N\, U f^{-1} \cos\theta\, B_{[1]} . && \text{(IV.107)}
\end{aligned}
$$

We have defined various quantities in order to make the above expressions more compact. These are

$$
\begin{aligned}
f(r,\theta) &= \sin^2\theta + U\cos^2\theta , \qquad m(r,\theta) = (U^{-1} + \cot^2\theta)^{-1} , \\
B_{[1]} &= d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2 . && \text{(IV.108)}
\end{aligned}
$$

Note that the metric has cohomogeneity two as it depends on the two coordinates r and θ. This is natural from the fact that the background (IV.105)-(IV.107) describes N D6-branes wrapped on a Kähler 4-cycle inside a CY_3; there is one transverse coordinate to the D6 inside the CY_3 and one which is in $\mathbb{R}^{1,3}$.

Before entering into the analysis of the moduli space, let us conclude by studying the range of validity of this solution. The supergravity approximation is valid where the curvature and the dilaton remain small. In this case, these restrictions imply

$$
\frac{1}{(g_sN)^{1/6}} \ll r \ll \frac{N^{1/4}}{g_s^{1/12}} . \qquad \text{(IV.109)}
$$

Within this range, this is the type IIA background which is proposed to be dual to the $\mathcal{N}=2$ $SU(N)$ SYM in 2+1, without any matter multiplet.

IV.9.3.2 The moduli space from supergravity

In this section we repeat the steps that led us to a zero-dimensional moduli space in the non-supersymmetric reduction considered above. We will analyze the Coulomb branch of this theory by giving a non-zero vacuum expectation value to the scalars in a $U(1)$ subgroup of $SU(N)$. As is well known, this is easily implemented in the supergravity side by pulling one of the N D6-branes away from the others. The $U(1)$ degrees of freedom on

the probe brane can be effectively described by the DBI action, where the rest of the branes are substituted by the background that they create.

If we want to break the gauge group without breaking supersymmetry, we must make sure that no potential is generated. So the first thing to look at is the vacuum configuration of the probe brane. With this purpose, we take the static gauge where the first seven space-time coordinates are identified with the worldvolume ones, and all the rest, *i.e.* $\{r, \theta, \psi\}$, are taken to be constant. In this way, only the potential is left in the DBI action. It is not possible to give a closed analytic expression for it but, numerically, it is easy to see that it vanishes only at $\theta = 0$ and $\theta = \pi$, independently of r and ψ.

We therefore locate the probe brane at such values of θ and look at the low energy effective action for its massless degrees of freedom. This is accomplished by allowing $\{r, \psi\}$ and the worldvolume field-strength $F_{[2]}$ to slowly depend on the worldvolume coordinates, so that only the terms quadratic in their derivatives are kept in the expansion of the DBI action. Indeed, in the limit in which the four-cycle is taken to be small and physics are $(2+1)$-dimensional, one can simply consider excitations about the flat non-compact part of the worldvolume. Both locus $\theta = 0, \pi$ give the same effective action:

$$-S_{probe} = \int d^3x \left[a^2 N (g_s N)^{3/2} C^2(r) r^2 \, (\partial r)^2 \; + \; \frac{1}{g_s N^2} \frac{1}{4C^2(r)} \, (\partial y)^2 \right] , \tag{IV.110}$$

where $a^2 = 2\pi^2 \mu_6$, $C(r) = \frac{1}{4}(r^2 + l^2)$ and y is the compact scalar of period 2π that one obtains after dualizing the gauge field $V_{[1]}$.

The moduli space is therefore two-dimensional and, after gluing the two locus at the origin, it turns out to have the topology of a cylinder. The metric is just

$$\begin{aligned} ds^2_{moduli} &= a^2 N (g_s N)^{3/2} C^2(r) \, r^2 \, dr^2 \; + \; \frac{1}{g_s N^2} \frac{1}{4C^2(r)} \, dy^2 \\ &= d\rho^2 + \frac{4a}{a g_s N^2 l^4 + 16 (g_s N^3)^{1/4} \, \rho} \, dy^2 \end{aligned} \tag{IV.111}$$

In the last step we redefined the radial coordinate $\rho = \frac{a g^{3/4} N^{5/4}}{16} \, r^2 \, (r^2 + 2l^2)$ in order to put the metric in a more standard form. It is easy to prove that this metric is Kähler by explicitly constructing the Kähler potential. In order to do so, first define complex coordinates

$$z = y + i\chi(r) \, , \tag{IV.112}$$

with

$$\chi(r) := \frac{a}{48} N (g_s N)^{5/4} \left(r^6 + 3l^2 r^4 + 3l^4 r^2 \right) . \tag{IV.113}$$

One can then show that

$$ds^2_{\text{moduli}} \; = \; 2\, g_{z\overline{z}}\, dz \otimes d\overline{z} \, , \tag{IV.114}$$

with $g_{z\overline{z}} = \partial_z \partial_{\overline{z}} \varphi$ and

$$\varphi = \frac{a^{2/3}}{128} g_s^{1/2} N^{5/6} \left(a g_s^{3/4} N^{5/4} l^6 + \frac{48}{g_s^{1/2} N} \, \frac{z - \overline{z}}{2i} \right)^{\frac{4}{3}} = \frac{a^2 N (g_s N)^{3/2}}{128} (r^2 {+} l^2)^4 \, . \tag{IV.115}$$

The fact that φ is a real function completes the proof that the metric is Kähler (see [107, 108] for similar results using different branes).

IV.9.4 Comparison with the field theory results

We shall now compare the results obtained using supergravity with the ones that are known from the field theory. The first immediate comment is that in the absence of matter multiplets, instantons of non-abelian gauge theories with $\mathcal{N} = 2$ in 2+1 dimensions develop a superpotential that completely lifts the Coulomb branch [109]. This is not in contradiction with our result because these contributions are exponentially suppressed with N, so they are not expected to be visible in the supergravity side.

On the other hand, $\mathcal{N} = 2$ supersymmetry implies that the moduli space must be a 2d Kähler manifold, in agreement with what we have seen from supergravity. Furthermore, it must have the topology of a cylinder, with the compact direction coming from the dualized scalar, and the non-compact one coming from the vacuum expectation value of the other scalar in the multiplet. The one loop corrected metric for an $SU(2)$ theory [110] is

$$ds^2 = \frac{1}{4} \left(\frac{1}{e^2} - \frac{2}{r} \right) dr^2 + \left(\frac{1}{e^2} - \frac{2}{r} \right)^{-1} dy^2 \, , \tag{IV.116}$$

and it is valid for $r \gg 2e^2$. Asymptotically, it tends to the classical prediction which, after generalizing to $SU(N)$, is just a flat cylinder with metric

$$ds^2 = \frac{1}{4e^2} dr^2 + \frac{e^2}{N} dy^2 \, . \tag{IV.117}$$

In order to compare these metrics with our supergravity result (IV.111) we shall perform a change of variables in (IV.116) so that the metric is

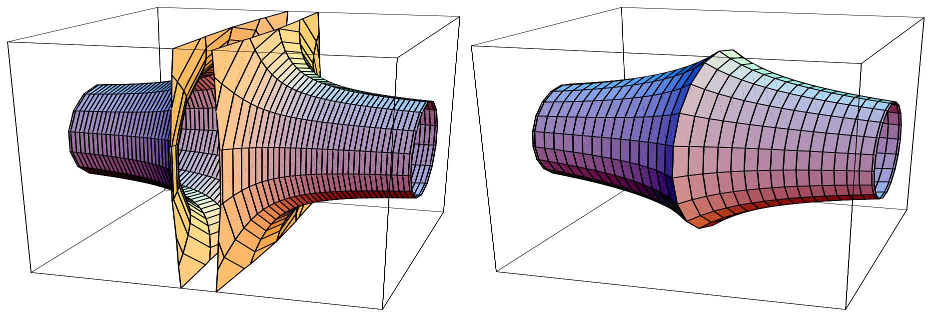


Fig. IV.4: Moduli spaces from 1-loop field theory (left) and supergravity (right).

in the standard form $ds^2 = d\rho^2 + f(\rho)dy^2$. Unfortunately, the change of variables is not expressible in terms of elementary functions. Anyway, we can solve numerically for $f(\rho)$ and plot the two moduli spaces, as we have done in figure IV.4. The plot on the left shows the one-loop corrected moduli space predicted by field theory calculations. At very large values of the non-compact scalar, it tends to flat cylinder with radius proportional to $|e|/\sqrt{N}$. As this *vev* decreases higher loop corrections are needed. In particular, the one loop calculation diverges at $r = 2e^2$.

On the other hand, the figure on the right shows the moduli space predicted by supergravity. It also tends to a cylinder with vanishing radius at large values of the non-compact scalar, so it qualitatively agrees with the $N \to \infty$ limit of the field theory. It also smooths the divergence of the one loop calculation, which could maybe correspond to a resummation of infinite loops contributions. Strictly speaking, we see from (IV.109) that the supergravity approximation is not valid at $r = 0$, where the curvature of our background blows up. In any case, we can still use it as close to the origin as needed by taking $g_s N$ large enough.

Finally, we shall make more explicit the relation between the parameters in supergravity (g_s, N and l) and in the field theory (e and N). As usual, the number N of D6-branes is the rank of the gauge group. On the other hand, in the supergravity side, a non-zero value for l prevents the radius from diverging as we approach the origin. Nevertheless, it is difficult to make the dictionary more precise. In any case, one can read the gauge coupling for the $U(1)$ degrees of freedom at a certain point of the moduli space by identifying the coefficient in front of the F^2 term in the DBI action of probe. The result is

$$\frac{1}{g^2_{U(1)}} = 4\pi^2 \mu_6 \, g_s N^2 \left(r^2 + l^2\right) . \qquad \text{(IV.118)}$$

V. FROM D-BRANES TO NC FIELD THEORIES

This chapter is devoted to the study of noncommutative field theories. We review the original motivation that led physicists to consider them, which dates back to more than 50 years. We take some time to review the Landau problem, and show that this is essentially the way that NC theories ultimately arise in String Theory. We then analyze the most remarkable classical and quantum properties that these theories have, paying special attention to the Seiberg-Witten map, the UV/IR mixing and the lack of unitarity in the electric cases. This last issue is most clearly discussed in section V.4, which contains some detailed one-loop computations that were reported in [41].

We leave for chapter VII the Hamiltonian analysis of non-local field theories and its application to the electric NC theories. The construction of string duals of magnetic NC theories and the non-perturbative physics than can be extracted from them is left for chapter VI.

V.1 The interest of NC field theories *per sé*

Despite the recent interest for NC theories originated by their appearance in string theory, the idea of extending the canonical position-momentum noncommutativity to the spacetime variables, represented by Hermitian operators on a Hilbert space such that

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}(x)\,, \tag{V.1}$$

is quite an old one. Apparently the first proposal along these lines is due to Heisenberg and dates back to the late 30s. He was hoping that such a modification could help curing the UV divergences typical of field theories. He probably mentioned this to Peierles who used it in a phenomenological approach to the study of electrons in external fields [111]. Then Peierles told about it to Pauli, who told Oppenheimer, who told to his student

Snyder, who published the first paper with a systematic analysis on the subject [112].

Commutation relations like (V.1) can be thought of an effective approach to whatever theory of quantum gravity rules Nature at the highest energies. For the simplest case of constant $\theta^{\mu\nu}$, this matrix plays a universal role analog to $\hbar$ for the spacetime coordinates and, in particular, it leads to uncertainty relations of the style

$$\Delta x^{\mu} \Delta x^{\nu} \geq |\, \theta^{\mu\nu} | \,. \tag{V.2}$$

It therefore predicts a minimal spacetime area that can be possibly probed, its size being of order θ. This strongly reminds the properties of some other aimed-to-be fundamental theories like:

- **String theory.** It is well-known that the effective Heisenberg principle that is relevant for strings takes the form

$$\Delta x \geq \frac{\hbar}{2} \left(\frac{1}{\Delta p} + l_s^2 \Delta p \right) \,. \tag{V.3}$$

  This implies, by minimization of the RHS with respect to the momentum, that the minimal spacetime length that can be measured is $\Delta x \sim l_s$.

- **κ-Minkowsky spacetimes.** This is an approach to Quantum Gravity in which two fundamental invariant scales are introduced. The first one is the usual speed of light c, which leads to the standard Poincaré algebra. The second one is the Planck length l_P which, as mentioned, is promoted to a length that is invariant for all inertial observers. This clearly does not affect the translational sector of the Poincaré subalgebra, but it completely modifies the boost sector, since the Lorentz contraction effect would spoil the invariance of l_P. Altogether, it requires a modification of the commutation relations among the spacetime coordinates to what is known as the algebra of κ-Minkowsky spacetime

$$[x^i, x^0] = i\, l_P\, x^i \,, \qquad [x^i, x^j] = 0 \,. \tag{V.4}$$

Any theory of quantum gravity has to propose how the spacetime notion must be modified at the shortest scales, and it seems hard to make compatible the principles of Quantum Mechanics with our familiar description of spacetime via differential manifolds. It is however not clear whether a

more proper description will arise in terms of some completely new mathematical construction, or whether the concept of a manifold will survive, but with modification of the standard commutation rules. In any case, the more humble aim of studying noncommutative spacetimes as toy models for an effective description seems reasonable and, as we will see, even the simplest infinitesimal deviation of θ from zero radically changes the quantum properties of the theory. This in turn will force us to deal with new unexpected physics.

Before finishing this section, we would like to review a familiar problem in standard physics in which space noncommutativity appears: the Landau problem. Apart from showing that noncommutativity is not a bizarre property of sophisticated limits in string theory, it will set up an intuitive understanding that will be needed in the following sections.

V.1.1 The Landau Problem

The Landau problem is possibly one of the simplest setups to describe the fundamental notions of noncommutativity. Consider the motion of N interacting non-relativistic electrons in a plane with a constant transverse magnetic field. Denote their positions and velocities by

$$\vec{r}_a = (x_a, y_a)\,, \qquad \vec{v}_a = \dot{\vec{r}}_a\,, \qquad a = 1, ..., N\,. \tag{V.5}$$

Pick a gauge where the vector potential is just $\vec{A}(\vec{r}_a) = (0, Bx_a)$, with $\vec{B} = \nabla \times \vec{A} = B\hat{z}$. The Lagrangian for the system is

$$L = \sum_a \left(\frac{1}{2} m_e \vec{v}_a^2 + \frac{e}{c} \vec{v}_a \dot{\vec{A}}(\vec{r}_a) - V(\vec{r}_a) \right) - \sum_{a<b} U(\vec{r}_a - \vec{r}_b)\,. \tag{V.6}$$

We have included a term V accounting for the possible interaction of the electron with the medium, and a sum of U-terms accounting for the pair-interaction among the electrons.

Let us perform the canonical quantization of the system. The canonical momenta are

$$\vec{p}_a = m_e \vec{v}_a + \frac{e}{c} \vec{A}(\vec{r}_a)\,, \tag{V.7}$$

and they satisfy the usual commutation relations

$$[x_a, p_b^x] = i\hbar\delta_{ab} = [y_a, p_b^y]\,, \qquad [x_a, y_b] = 0 = [p_a^x, p_b^y]\,. \tag{V.8}$$

Recall that the canonical momenta are not gauge invariant quantities, as it is obvious from (V.7). The gauge invariant, and therefore physical, quantities

are $\vec{\pi}_a = m_e \vec{v}_a$, and these momenta satisfy a noncommutative algebra

$$[\pi_a^x, \pi_b^y] = i\hbar \frac{eB}{c} \delta_{ab} \,. \tag{V.9}$$

The Hamiltonian for the system is

$$H = L = \sum_a \left(\frac{\vec{\pi}_a^2}{2m_e} + V(\vec{r}_a) \right) + \sum_{a<b} U(\vec{r}_a - \vec{r}_b) \,. \tag{V.10}$$

For $U = V = 0$, we can write the physical momenta in terms of annihilation/creation operators and one finds that the energy levels are

$$E = \sum_a \hbar\omega_c (n_a + \frac{1}{2}) \,, \qquad n_a = 0, 1, 2, ... \tag{V.11}$$

where

$$\omega_c = \frac{eB}{m_e c} \,, \tag{V.12}$$

is the cyclotronic frequency. This is the famous expression for the *Landau levels*. Note for future reference that the energy gap Δ between the ground state and the first excited ones is $\Delta = \hbar\omega_c/2$.

V.1.2 Projecting to the first Landau level

We would now like to take a limit in the parameter space of the Landau problem that will reappear (disguised) later when we deal with string theory. The limit can be thought either as a limit of *strong magnetic field* or of *small electron mass*. The dimensionless quantity controlling it is the quotient B/m_e which we take to be very large. In this limit, the Lagrangian (V.6) simplifies to

$$L \approx \sum_a \left(\frac{eB}{c} x_a \dot{y}_a - V(\vec{r}_a) \right) - \sum_{a<b} U(\vec{r}_a - \vec{r}_b) \,. \tag{V.13}$$

Note that we are dealing now with a first order Lagrangian, which is therefore singular. This means that we will get primary constraints that will not allow us to isolate all velocities in terms of the momenta. For example, the canonically computed momenta are now

$$p^x = 0 \,, \qquad p^y = \frac{eB}{c} x \,, \tag{V.14}$$

which do not involve the velocities and should therefore be treated as constraints in the Hamiltonian formalism.

One can now run the whole machinery of Dirac formalism for quantization in the presence of constraints. First, since the Poisson bracket of the constraints is non-zero they are second-class constraints. This allows one to solve them once and for all if one replaces all Poisson brackets by Dirac brackets, which are the ones that are finally promoted to commutators in the quantized theory. Having done this, one finds that the (reduced) phase space of the theory is two-dimensional; both momenta have been solved, and one is left with the following canonical structure for the remaining coordinates

$$[x_a, y_b] = i\frac{\hbar c}{eB}\delta_{ab}\,, \tag{V.15}$$

which can be written in the standard form

$$[x^i_a, x^j_b] = i\theta^{ij}\delta_{ab}\,, \qquad \theta^{ij} = \frac{\hbar c}{eB}\epsilon^{ij}\,. \tag{V.16}$$

Can we understand what is going on in the limit just taken? The first point to notice is that in this limit the energy gap Δ is scaled to infinity, so that the ground state is decoupled from the rest of excited states. This limit is such that the phase space is reduced, as we have actually seen; indeed, the constraints on the momenta project the allowed states of the system to its ground state. Moreover, the theory becomes topological in the sense that the Hamiltonian (V.10) reduces to

$$H_0 \approx \sum_a V(\vec{r}_a) + \sum_{a<b} U(\vec{r}_a - \vec{r}_b)\,, \tag{V.17}$$

and it vanishes in the absence of potentials, showing that there are no propagating degrees of freedom.

V.1.3 Weyl Quantization

Having established the first simple example of an algebra of quantum operators in which coordinates do not commute, it is useful to switch back to the classical mechanics phase space and look for an alternative formalism that takes noncommutativity into account. The method known as Weyl quantization will do the work for us. The general idea is to define a map from the algebra of quantum operators on a Hilbert space to the algebra of *functions* on the classical phase space. In what follows, in order to make

clear the distinction between classical functions and operators, we will use the standard convention of putting hats over the latter.

The first ingredient we need a proposal for a map between such functions and operators. Let us consider the so-called Weyl map defined by

$$W : F(x,p) \longrightarrow \hat{O}_F = W[F] := \frac{1}{2\pi}\int d\alpha d\beta f(\alpha,\beta) e^{i(\alpha\hat{x}+\beta\hat{p})}\,, \qquad \text{(V.18)}$$

where $f(\alpha,\beta)$ is the Fourier transform of $F(x,p)$. The second ingredient needed is a set of commutation relations among the quantum coordinate and momentum operators. This is necessary in order to define the product of two functions of the quantum phase-space operators, as can be seen by computing

$$W[F]W[G] = \frac{1}{(2\pi)^2}\int d\alpha d\beta \int d\alpha' d\beta' f(\alpha,\beta) g(\alpha',\beta') e^{i(\alpha\hat{x}+\beta\hat{p})} e^{i(\alpha'\hat{x}+\beta'\hat{p})}\,. \qquad \text{(V.19)}$$

We need to know how to evaluate the product of the exponentials in order to find whether the RHS is the W-image of some function $H(p,q)$. What is clear is that unless we choose all commutators among coordinate and momenta to be zero, it will follow that $W[F]W[G] \neq W[FG]$ and therefore the Weyl map will not preserve the usual product of functions on phase space. Let us show two of the main examples:

- If we take the usual quantum mechanics relations

$$[\hat{x},\hat{x}] = 0\,, \qquad [\hat{p},\hat{p}] = 0\,, \qquad [\hat{x},\hat{p}] = i\hbar\,, \qquad \text{(V.20)}$$

then we can use in (V.19) the Baker-Campbell-Hausdorff (BCH) expansion

$$e^A e^B = exp\left(A + B + \frac{1}{2}[A,B] + \frac{1}{12}[[A,B],B] + \frac{1}{12}[[B,A],A] + ...\right)\,, \qquad \text{(V.21)}$$

and obtain

$$\begin{aligned} W[F]W[G] &= \frac{1}{(2\pi)^2}\int d\alpha d\beta \int d\alpha' d\beta' f(\alpha',\beta') g(\alpha'-\alpha,\beta'-\beta) \times \\ &\times \; e^{i(\alpha\hat{x}+\beta\hat{p})} e^{\frac{i}{\hbar}(\alpha\beta'-\alpha'\beta)} = \frac{1}{2\pi}\int d\alpha d\beta\, a(\alpha,\beta)\, e^{i(\alpha\hat{x}+\beta\hat{p})}\,, \end{aligned}$$

where

$$a(\alpha,\beta) = \frac{1}{2\pi}\int d\alpha' d\beta' f(\alpha',\beta') g(\alpha'-\alpha,\beta'-\beta) e^{\frac{i}{\hbar}(\alpha\beta'-\alpha'\beta)}\,. \qquad \text{(V.22)}$$

This immediately forces us to define a new product $*$ for functions in the classical phase space by requiring that

$$(F * G)(x,p) = W^{-1}\left(W(F)W(G)\right) = \frac{1}{2\pi}\int d\alpha d\beta\, a(\alpha,\beta)e^{i(\alpha x+\beta p)}\,. \tag{V.23}$$

So all we need to know is to inverse-Fourier transform of (V.22), which is easily verified to give the following differential expression

$$(F * G)(x,p) = F(x,p)\exp\frac{i}{2}\hbar\left(\overleftarrow{\partial_x}\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\overrightarrow{\partial_x}\right)G(x,p)\,. \tag{V.24}$$

It can be verified as well that this product is associative but noncommutative. As a final check note that, as expected,

$$[x,p]_* \equiv x*p - p*x = i\hbar\,. \tag{V.25}$$

- We now wish to consider how the Weyl map works for the commutation relations that we obtained in the Landau problem (and that we will also obtain from the string theory low energy dynamics). This will lead us to the concept of *classical* noncommutative spaces over which fields will be defined. We therefore require a deformation of (V.20) that includes a spacetime noncommutativity with constant θ:

$$[\hat{x}^\mu,\hat{x}^\nu] = i\theta^{\mu\nu}\,, \qquad [\hat{p}^\mu,\hat{p}^\nu] = 0\,, \qquad [\hat{x}^\mu,\hat{p}^\nu] = i\hbar\,\eta^{\mu\nu}\,. \tag{V.26}$$

The strategy here is to first define the concept of *noncommutative spacetime* by Weyl-mapping only the coordinate sector, *i.e.* not including the momenta. One then has the capability of defining functions or fields in such spacetimes and, in particular, one can write down actions for such fields by means of whatever noncommutative product we obtain. At the end, if wanted, one can proceed by the usual quantization of such classical theories which, after all, takes into account the usual coordinate-momentum noncommutativity.

So, we set $\beta = 0$ in the definition of the Weyl map (V.18)

$$W : F(x) \longrightarrow W_\theta[F] := \frac{1}{(2\pi)^4}\int d^4\alpha f(\alpha)e^{i\alpha_\mu\hat{x}^\mu}\,, \tag{V.27}$$

and we compute the product of operators using the first commutator in (V.26). The steps are analog, and indeed simpler, to the previous case considered. The final answer is then

$$W_\theta[F]W_\theta[G] = W_\theta[F*G]\,, \tag{V.28}$$

with

$$(F * G)(x) = F(x) \exp \frac{i}{2}\theta^{\mu\nu} \left(\overleftarrow{\partial_{x^\mu}}\overrightarrow{\partial_{x^\nu}} - \overleftarrow{\partial_{x^\nu}}\overrightarrow{\partial_{x^\mu}}\right) G(x) \,. \tag{V.29}$$

This is the so-called Weyl-Moyal product that we will be using very often throughout the first part of this thesis. Again one can check that this product is associative but noncommutative, and that

$$[x^\mu, x^\nu]_* = i\theta^{\mu\nu} \,. \tag{V.30}$$

V.1.4 A few properties of the Weyl-Moyal product

Because of the relevance of the Moyal product (V.29), it is worth making a pause and collecting some of the main properties that will be needed when dealing with NC field theories.

Most properties follow from examining the expansion of the $*$-product of any two functions

$$f * g = fg + \frac{i}{2}\theta^{\mu\nu}\partial_\mu f \partial_\nu g - \frac{1}{4}\theta^{\mu\nu}\theta^{\alpha\beta}\partial_\mu\partial_\alpha f \partial_\nu\partial_\beta g \, + \, \dots \,. \tag{V.31}$$

Because of the antisymmetry and the constancy of the matrix θ, all terms but the first one can be written as a total derivative. For example, for the terms considered in (V.31),

$$f * g = fg + \partial_\mu \left(\frac{i}{2}\theta^{\mu\nu} f \partial_\nu g - \frac{1}{4}\theta^{\mu\nu}\theta^{\alpha\beta}\partial_\alpha f \partial_\nu \partial_\beta g\right) \, + \, \dots \,. \tag{V.32}$$

Immediate consequences affect integrals of $*$-products with boundary conditions such that surface terms vanish:

$$\int d^4x F * G \;\; = \;\; \int d^4x \, FG \,, \tag{V.33}$$

$$\int d^4x F_1 * F_2 * \dots * F_n \;\; = \;\; \int d^4x \, F_n * F_1 * \dots * F_{n-1} \,. \tag{V.34}$$

The first property will imply that the free part of the NC field theory actions will coincide with the commutative one: propagators will not change. The second cyclic property will imply that vertices in Feynman diagrams for NC theories will be invariant under a cyclic rotation of the incoming legs.

The last useful property we will review has to do with the way that plane waves are treated under $*$-products. It is not hard to check that

$$e^{ik_1x} * e^{ik_2x} * \dots * e^{ik_nx} = e^{ix\sum_i k_i - \frac{i}{2}\sum_{i<j} k_i\theta k_j} \,, \tag{V.35}$$

which will allow us to write the Fourier transform of interactions by means of

$$\int d^4x F_1 * F_2 * ... * F_n = \frac{1}{(2\pi)^{n-1}} \int \prod_{i=1}^{n} \left[d^4 p_i f_i(p_i)\right] \ e^{-\frac{i}{2}\sum_{j<l} k_j \theta k_l} \, . \quad \text{(V.36)}$$

V.2 From D-branes to NC theories

We now wish to explain how NC theories arise in string theory as originally discovered by Seiberg and Witten in [25]. The final goal is to show that the low energy worldvolume effective action of D-branes in the presence of a constant background B-field is a NC theory.

Since D-brane dynamics are described in perturbation theory by the excitations of the open strings that lie on them, let us start by considering the worldsheet 2d CFT for such strings. The bosonic part of the action in conformal gauge is

$$S = \frac{1}{4\pi\alpha'} \int_{\Sigma_2} \left(g_{MN} \partial_\alpha X^M \partial^\alpha X^N - 2\pi i \alpha' B_{MN} \epsilon^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N \right) . \quad \text{(V.37)}$$

Some clarification on the conventions:

- Spacetime $\longleftrightarrow$ Capital indices $M, N = 0, ..., D-1$.
- Worldsheet $\longleftrightarrow$ Greek indices $\alpha, \beta = \{\tau, \sigma\}$.
- The worldsheet metric is Euclidean, and hence the i in front of the second term.
- We will divide the spacetime indices among indices parallel $(\mu, \nu ...)$ and transverse $(i, j, ...)$ to the brane.

Note that since we take the background B-field to be constant, the second term of the action can be written as a total derivative

$$S[B] = i \int_{\Sigma_2} P[B_2] = \frac{i}{2} \int_{\Sigma_2} P\left[d(B_{\mu\nu} X^\mu dX^\nu) \right] = \frac{i}{2} \int_{\partial\Sigma_2} B_{\mu\nu} X^\mu \partial_t X^\nu \, , \quad \text{(V.38)}$$

where we recall that P is the pullback operator and ∂_t stands for the derivative tangent to the boundary $\partial\Sigma_2$.

The background we want to deal with has

$$g_{MN} = \eta_{MN} \, , \qquad B_{\mu\nu} = \text{ct.} \, , \; B_{ij} = 0 = B_{\mu i} \, , \qquad \Phi = \text{ct.} \quad \text{(V.39)}$$

It is very important to remark here that a *constant B-field would be pure gauge if it was not for the presence of D-branes.* When branes are present, trying to gauge away a B-field with components along the brane induces a worldvolume field-strength F_2 since the truly invariant quantity is then $\mathcal{F}_2 = B_2 + 2\pi\alpha' F_2$. In other words, we can always go to a gauge where all

the components of the constant B-field transverse to the brane directions are zero. On the other hand, the gauge ambiguity on how to distribute the value of $\mathcal{F}_2$ among F_2 and B_2 is commonly resolved by the adhoc boundary condition that F_2 vanishes at infinity. This relation between B_2 and F_2 motivates the name *electric* for the $B_{0\mu}$ components and *magnetic* for the B_{ij} ones.

Having fixed the background we vary the action to obtain the equations of motion and the boundary conditions. Since the B-term in the Lagrangian is a total derivative, the equations of motion are the usual ones; however the requirement that the surface terms vanish is modified to

$$\begin{aligned} \partial_n X_\mu + 2\pi i\alpha' B_{\mu\nu}\partial_t X^\nu|_{\partial\Sigma_2} &= 0\,, \qquad \mu = 0, ..., p\,, &\text{(V.40)}\\ \partial_t X^i|_{\partial\Sigma_2} &= 0\,, \qquad i = p+1, ..., 9\,. &\text{(V.41)} \end{aligned}$$

Note that the effect of the B-field is to interpolate between Neumann BC's at weak field and Dirichlet BC's at strong field. In the latter limit, which is the one we will take in a while, the endpoints of the strings are fixed at one point of the worldvolume of the brane, as if they were attached to a D0 brane.

We will now move on an try to compute S-matrix elements with this CFT with the aim to extract its low energy physics and see what kind of field theory approximates them. As we are interested in the physics of the D-branes, we will mostly be concerned with open string diagrams. Let us start by tree level diagrams with the topology of a disc and map it to the upper half plane by a conformal transformation. The BC's (V.40) read in the usual complex coordinates

$$(\partial - \bar\partial)X_\mu + 2\pi i\alpha' B_{\mu\nu}(\partial + \bar\partial)X^\nu|_{z=\bar z} = 0\,. \qquad \text{(V.42)}$$

The propagator satisfying these boundary conditions is

$$\begin{aligned} \langle X^\mu(z)X^\nu(z')\rangle &= -\alpha'\big[\eta^{\mu\nu}\log|z-z'| - \eta^{\mu\nu}\log|z-\bar z'| &\text{(V.43)}\\ &\quad + G^{\mu\nu}\log|z-\bar z'|^2 + \frac{1}{2\pi\alpha'}\theta^{\mu\nu}\log\frac{z-\bar z'}{\bar z - z'} + D^{\mu\nu}\big]\,, \end{aligned}$$

where

$$\begin{aligned} G^{\mu\nu} &= \left(\frac{1}{\eta+2\pi\alpha' B}\eta\frac{1}{\eta-2\pi\alpha' B}\right)^{\mu\nu}\,, &\text{(V.44)}\\ G_{\mu\nu} &= \eta_{\mu\nu} - (2\pi\alpha')^2\big(B\eta^{-1}B\big)_{\mu\nu}\,, &\text{(V.45)}\\ \theta^{\mu\nu} &= -(2\pi\alpha')^2\left(\frac{1}{\eta+2\pi\alpha' B}B\frac{1}{\eta-2\pi\alpha' B}\right)^{\mu\nu}\,. &\text{(V.46)} \end{aligned}$$

and the constants D^{ij} can depend on B but not no z and z', and therefore play no essential role.

If we are only interested in the interactions of open strings with other open strings, all vertex operators that we need to insert in the path integrals must lie in the boundary of Σ_2. So, effectively, we just need to know the propagator (V.43) restricted to the boundary. Naming $\tau = \mathrm{Re}[z]$, we get

$$\langle X^\mu(\tau) X^\nu(\tau')\rangle = -\alpha' G^{\mu\nu} \log(\tau - \tau')^2 + \frac{i}{2}\theta^{\mu\nu}\epsilon(\tau - \tau')\,.$$

We are ready to give a physical interpretation to the two metrics that have appeared at this point. Whereas the short distance behavior of the propagator (V.43) is controlled by the term $-\alpha'\eta^{\mu\nu}\log|z-z'|$, the corresponding leading term for points in the boundary is controlled by $-\alpha' G^{\mu\nu}$-$\log|z-z'|$. Recall that one way to obtain the mass-shell condition for the string states is to impose that its corresponding vertex operator has dimension one. This in turn can be calculated via its OPE with the energy momentum tensor, and it is easy to see that the relevant singular terms are precisely the ones containing the logarithms; the metric multiplying them is the one that has to be used when computing the mass. Therefore we can say that the closed strings (whose vertex operators are inserted inside the disc) still feel the presence of a flat background metric, whereas open strings effectively see $G_{\mu\nu}$. Henceforth, the former will be called the *closed string metric* and the latter, the *open string metric*. As a consequence, the correct vertex operators for the tachyon and the massless gauge field of the open string spectrum are

$$V_{tachyon}(k) \sim :e^{ik\cdot X}:\,, \qquad V_{gauge}(k,\xi) \sim :\xi\cdot\partial\, e^{ik\cdot X}: \tag{V.47}$$

where the dot-contractions are taken with respect to the open string metric G. In particular, the gauge boson vertex operator with polarization ξ is physical when

$$G^{\mu\nu}\xi_\mu k_\nu = 0 = G^{\mu\nu}k_\mu k_\nu\,. \tag{V.48}$$

The tensor $\theta^{\mu\nu}$ in (V.47) has also an interpretation. By the usual calculus in CFT, commutators of operators are translated into time-ordered products, so that

$$[X^\mu(\tau), X^\nu(\tau)] = \langle X^\mu(\tau)X^\nu(\tau-\epsilon) - X^\mu(\tau)X^\nu(\tau+\epsilon)\rangle = i\theta^{\mu\nu}\,. \tag{V.49}$$

Therefore, the open strings feel that they are effectively living on a noncommutative space with parameters $\theta^{\mu\nu}$.

V.2.1 The low energy limit for magnetic backgrounds

Our aim now is to establish what the low energy effective theory is. We want to decouple all the massive states, which as usual should be accomplished by taking $\alpha' \to 0$. However we will see that this is not possible to do in the cases for which there are non-zero electric components $B_{0\mu} \neq 0$. We therefore restrict in this section to magnetic backgrounds and postpone the discussion of electric ones until section V.4.1.2.

The limit $\alpha' \to 0$ must be taken with care if we want to remain with a sensible theory for open strings; we need to keep finite the parameters that control their dynamics, *i.e.* $G_{\mu\nu}$ and $\theta^{\mu\nu}$. Taking a look at the formulas (V.44)-(V.46) we see that this can be achieved by taking $\alpha' B \gg 1$, which is a *strong magnetic field limit*. In the original paper of Seiberg and Witten, instead of sending the magnetic field to infinity, they equivalently scaled to zero the closed string metric η, keeping B fixed. The exact way in which the limit is taken is then

$$\alpha' \sim \epsilon^{\frac{1}{2}} \to 0\,, \qquad \eta_{\mu\nu} \sim \epsilon \to 0\,. \tag{V.50}$$

Then, the relations (V.44)-(V.46) reduce to

$$\begin{aligned} G^{\mu\nu} &= -\frac{1}{(2\pi\alpha')^2}\left(\frac{1}{B}\eta\frac{1}{B}\right)^{\mu\nu}, &\text{(V.51)}\\ G_{\mu\nu} &= -(2\pi\alpha')^2(B\eta^{-1}B)_{\mu\nu}\,, &\text{(V.52)}\\ \theta^{\mu\nu} &= \left(\frac{1}{B}\right)^{\mu\nu}. &\text{(V.53)}\end{aligned}$$

Using (V.50) we see that G and θ are finite in the limit. Let us analyze how other quantities look like in the limit. The propagator (V.47) becomes

$$\langle X^\mu(\tau)X^\nu(0)\rangle = \frac{i}{2}\theta^{ij}\epsilon(\tau)\,. \tag{V.54}$$

Note that it looses its α'-dependence; a sign that we are decoupling all the massive modes of the string. Indeed, all its dependence on the worldsheet coordinates is through $\epsilon(\tau)$, which is a constant function everywhere except at one point. In a sense, the CFT looses all its propagating degrees of freedom. This is clearly seen by taking a look at the action (V.37) in this limit

$$S \to -\frac{i}{2}\int_{\partial\Sigma_2} B_{\mu\nu}X^\mu\partial_t X^\nu\,. \tag{V.55}$$

We see that the kinetic term in the bulk has become negligible, and that we are left with a one-dimensional action for two oppositely charged particles

in a large constant magnetic field. This is precisely the Landau problem that we considered in V.1.1! We can therefore understand better why the noncommutativity (V.49) appears, since position operators are now canonical pairs describing electrons forced to remain in its Landau ground state. It is worth mentioning that the fact that our two electrons are indeed the endpoints of a string makes a little difference in this guise, specially when one compactifies some direction, say, on a torus; there we can see global effects arising. However, for the most part of what follows, the discussed interpretation remains a useful guide.

V.2.2 The effective action from the S-matrix

Let us setup the technicalities that we will need in order to compute S-matrix elements and deduce the effective action. The analog of the $*$-product that we found by Weyl mapping in (V.1.3) is provided here by normal ordering. By using repeatedly the operator product expansion of two exponentials one can prove the following properties:

1. $: e^{ip\cdot X(\tau)} :: e^{iq\cdot X(0)} := e^{-\frac{i}{2}\epsilon(\tau)(p\theta q)} : e^{ip\cdot X(\tau)+iq\cdot X(0)} :$,
2. $: f(X(\tau)) :: g(X(0)) :=: e^{\frac{i}{2}\epsilon(\tau)\theta^{\mu\nu}\frac{\partial}{\partial X^{\mu}(\tau)}\frac{\partial}{\partial X^{\nu}(0)}} f(X(\tau))g(X(0)) :$
3. $\lim_{\tau\to 0^+} : f(X(\tau)) ::: g(X(0)) :=: (f * g)(X(0)) :$,

with the $*$-product defined in (V.29). From these properties it follows that the correlation functions of open string tachyon vertex operators (V.47) on $\partial\Sigma_2$ are given by

$$\langle\prod_n : e^{ip^n\cdot X(\tau_n)} :\rangle = e^{-\frac{i}{2}\sum_{n>m} p^n\cdot\theta\cdot p^m}\delta(\sum_n p^n)\,. \tag{V.56}$$

It is worth pausing a minute to examine this equation. There a two main difference with respect to the same disc computations made in a background without B-fields. The first one is, as mentioned, the appearance of the open string metric G in the various scalar products. The second one is the appearance of a momentum dependent phase. By the $*$-product cyclic property (V.34), we see these correlation functions are invariant under cyclic permutations of the operators on the boundary, although *they are not invariant under non-cyclic permutations.* This resembles very much the non-abelian properties of open strings with Chan-Paton factors, although we remark

that we have not yet introduced any. As we will see, abelian noncommutative actions[1] share a lot of properties of standard non-abelian theories. In particular, the lost of cyclicity will lead to the classification of diagrams in terms of planar and non-planar ones.

Equation (V.56) can be rewritten in terms of $*$-products since they are the natural operation for the problem. Indeed, using the property (V.35) in (V.56) we get

$$\langle \prod_n : e^{ip^n \cdot X(\tau_n)} : \rangle = \int dx \, e^{ip^1 \cdot x} * e^{ip^2 \cdot x} * \ldots * e^{ip^n \cdot x} \,. \tag{V.57}$$

More generally, all expectation values made of the scalar fields (with no derivatives) can be shown to give

$$\left\langle \prod_n : f_n(X(\tau_n)) : \right\rangle = \int dx (f_1 * f_2 * \cdots * f_n)(x) \,. \tag{V.58}$$

Equations (V.56)-(V.58), together with their extension to include derivatives of the scalar fields (which just follow by the usual procedure) are all we need to compute our first S-matrix element: the 3 gauge bosons interaction. An intermediate result is

$$\begin{aligned} \left\langle \xi^1 \cdot \partial x e^{ip^1 \cdot x(\tau_1)} \, \xi^2 \cdot \partial x e^{ip^2 \cdot x(\tau_2)} \, \xi^3 \cdot \partial x e^{ip^3 \cdot x(\tau_3)} \right\rangle \sim \tfrac{1}{(\tau_1-\tau_2)(\tau_2-\tau_3)(\tau_3-\tau_1)} \\ \cdot \left(\xi^1 \cdot \xi^2 p^2 \cdot \xi^3 + \xi^1 \cdot \xi^3 p^1 \cdot \xi^2 + \xi^2 \cdot \xi^3 p^3 \cdot \xi^1\right) \\ \cdot e^{-\frac{i}{2}\left(p^1 \cdot \theta \cdot p^2 \epsilon(\tau_1-\tau_2) + p^2 \cdot \theta \cdot p^3 \epsilon(\tau_2-\tau_3) + p^3 \cdot \theta \cdot p^1 \epsilon(\tau_3-\tau_1)\right)} \,, \end{aligned} \tag{V.59}$$

where we are not writing an explicit delta-function for the momentum conservation. To compute the S-matrix element, we should integrate over all possible positions of the vertex operators in $\partial\Sigma_2$, and then divide over the volume of the conformal group of the disc $SL(2,R)$. Equivalently, for this 3-point amplitude $SL(2,R)$ is large enough as to allow us to place the vertex operators wherever we like (typically, at $\{0,1,\infty\}$). In any way we make it, the result is just equation (V.59) removing the τ-dependent denominator, and with a little modification of the exponential

$$\left(\xi^1 \cdot \xi^2 p^2 \cdot \xi^3 + \xi^1 \cdot \xi^3 p^1 \cdot \xi^2 + \xi^2 \cdot \xi^3 p^3 \cdot \xi^1\right) e^{-\frac{i}{2}(p^1 \cdot \theta \cdot p^2)} \,. \tag{V.60}$$

This amplitude can be reproduced by a computation of the 3-point function evaluated with the field theoretical action

$$\frac{(\alpha')^{\frac{3-p}{2}}}{4(2\pi)^{p-2} G_s} \int d^{p+1}x \, \sqrt{\det G} G^{\mu\mu'} G^{\nu\nu'} \mathrm{Tr} \; \hat{F}_{\mu\nu} * \hat{F}_{\mu'\nu'} \,, \tag{V.61}$$

[1] We recall that in the whole thesis we use 'abelian' to refer to gauge groups and 'commutative' to refer to spacetime commutativity.

where G_s will is what we will call the open string coupling (we will fix its value later) and

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i\hat{A}_\mu * \hat{A}_\nu + i\hat{A}_\nu * \hat{A}_\mu \,. \tag{V.62}$$

We are advancing what will follow a little bit by putting hats over all NC-fields, not to confuse them with the commutative ones (both concepts, will be defined shortly).

We finish this section by generalizing the result to include any number of gauge fields. It can be seen that

$$\left\langle \prod_n \xi^n \cdot \partial x e^{ip^n \cdot x(\tau_n)} \right\rangle_{G,\theta} = e^{-\frac{i}{2}\sum_{n>m} p^n \cdot \theta \cdot p^m \epsilon(\tau_n - \tau_m)} \left\langle \prod_n \xi^n \cdot \partial x e^{ip^n \cdot x(\tau_n)} \right\rangle_{G,\theta=0} . \tag{V.63}$$

In other words, if the effective action is expressed in terms of the open string quantities, all the θ-dependence is only in the $*$-product.

V.2.3 A look at the NC Yang-Mills action and NC gauge invariance

The action (V.61) is the first NC field theory that we encounter. A little bit expectedly, it is not obtained from an ordinary U(1) YM action simply by replacing standard products by $*$-products; the field strength (V.62) has been non-linearized as well. Indeed, equation (V.62) defines a NC field strength exactly in the same way as we would define a non-abelian one. Further more, the action is not invariant under usual gauge transformations $\delta A_\mu = \partial_\mu \lambda$ but rather under a NC version of them

$$\delta \hat{A}_\mu = \partial_\mu \lambda + i[\lambda, \hat{A}_\mu]_* \,. \tag{V.64}$$

The need for this gauge transformation can be understood directly from string theory. In the presence of a B-field, the coupling of the gauge field $A_\mu(X)$ to the worldsheet

$$-i \int d\tau A_\mu(x) \partial_\tau X^\mu \,, \tag{V.65}$$

is no longer invariant under $\delta A_\mu = \partial_\mu \lambda$ at the quantum level. This is because the gauge transformation of (V.65), which is a total derivative $\int d\tau \partial_\tau \lambda$, can produce divergences when meeting other insertions in the path integral. For example, a term like

$$\int d\tau A_\mu(x) \partial_\tau X^\mu \int d\tau' \partial_{\tau'} \lambda \,, \tag{V.66}$$

appears in the variation of the path integral, and needs to be regularized at points where the operators are very close. Seiberg and Witten showed that point splitting regularization produces a finite contribution which can only be cancelled if the full gauge transformation contains the extra piece in (V.64). Due to the fact that point splitting explicitly breaks gauge invariance, a modification was to be expected. However, what if we use a regularization compatible with gauge invariance? Had we used Pauli-Villars, the original transformation would have remained an invariance. Conclusion: getting a NC field $\hat{A}$ or an ordinary one A (the adjective referring to their kind of gauge transformation) is a matter of choice of regularization method.

Are we then just being masochists by choosing $*$-products and NC fields? The answer to this question is arguably *no*. The freedom in choosing commutative or NC fields corresponds to the well-known ambiguity in low energy Lagrangians derived from string S-matrices. The S-matrix is unchanged under field redefinitions of the effective Lagrangian (V.61). We could have well written an analog Lagrangian in terms of the closed string quantities $\{\eta_{\mu\nu}, g_s, B_2\}$ and the commutative fields A_μ reproducing the same S-matrix element. For example, had we used Pauli-Villars in the worldsheet, the commutative effective action would have been an ordinary U(1) Maxwell Lagrangian replacing F_2 by $F_2 + B_2$ everywhere, as the latter is the only gauge-invariant quantity and PV preserves gauge invariance. However, computations are carried more easily with the NC action (V.61) because all the dependence on θ is hidden inside the $*$-product, which is quite a manageable structure.

Let us finish this section by fixing the value of G_s. This can be done by comparing the DBI actions that one would obtain using closed or open string variables. As already discussed in sec. II.1.2 the DBI action in the presence of B-fields (which we now reinterpret as being written in the closed string variables) is

$$\mathcal{L}_{DBI}(F) = \frac{1}{g_s (2\pi)^p (\alpha')^{\frac{p+1}{2}}} \sqrt{\det(g + 2\pi\alpha'(B+F))} \tag{V.67}$$

whereas the NC counterpart must have its θ-dependence in the $*$-product and it is thus expected to be

$$\mathcal{L}_{DBI}(\hat{F}) = \frac{1}{G_s (2\pi)^p (\alpha')^{\frac{p+1}{2}}} \sqrt{\det(G + 2\pi\alpha' \hat{F})_*}\,. \tag{V.68}$$

All products in any expansion of (V.68) are understood to be $*$-products.

Comparing the coefficients of both Lagrangians for constant gauge fields

$$\mathcal{L}_{DBI}(F=0) = \mathcal{L}_{DBI}(\hat{F}=0)\,, \tag{V.69}$$

we obtain the required value for the open string coupling G_s before and after the zero slope limit:

$$\begin{aligned} G_s &= g_s \left(\frac{\det G}{\det(g+2\pi\alpha' B)}\right)^{\frac{1}{2}} = g_s \left(\frac{\det G}{\det g}\right)^{\frac{1}{4}} \\ &= g_s \left(\frac{\det(g+2\pi\alpha' B)}{\det g}\right)^{\frac{1}{2}}\,, \end{aligned} \tag{V.70}$$

$$G_s|_{\alpha'\to 0} = g_s \text{det'}(2\pi\alpha' B g^{-1})^{\frac{1}{2}}\,, \tag{V.71}$$

where det' stands for the determinant of the non-vanishing block of B.

We can now obtain the NC Yang-Mills coupling that originates from expanding the NC DBI action (V.68) and picking the coefficient of the $\hat{F}^2$ term:

$$\frac{1}{g_{YM}^2} = \frac{(\alpha')^{\frac{3-p}{2}}}{(2\pi)^{p-2} G_s}\,. \tag{V.72}$$

In order to keep it finite in the zero slope limit, we need to scale G_s exactly in the same way as one would scale g_s in the $B=0$ cases [51], *i.e.* $G_s \sim \epsilon^{\frac{3-p}{4}}$. The complete zero slope limit is then, in terms of closed string quantities,

$$\alpha' \sim \epsilon^{\frac{1}{2}} \to 0\,, \qquad \eta_{\mu\nu} \sim \epsilon \to 0\,, \qquad g_s \sim \epsilon^{\frac{3-p+r}{4}}\,, \tag{V.73}$$

where r is the rank of B_2.

Before finishing this section it is worth noticing a subtle fact. Whereas we know that the group $U(N)$ is actually isomorphic to $U(1)\times SU(N)/Z_N$, so that the $U(1)$ photon decouples from the rest of degrees of freedom in a $U(N)$ gauge theory, this does not apply to NC theories. If f and g are two matrix valued fields, we have that in general $\det(f*g) \neq \det(f)*\det(g)$ so that $SU(N)$ does not give rise to any gauge group on a NC $\mathbb{R}^D$. In other words, the $U(1)$ photon of NC $U(N)$ gauge theories couples to the rest of fields in the gauge multiplet.

V.2.4 The Seiberg-Witten map

As we have seen, different regularizations in the worldsheet lead to different effective YM theories, a commutative and a NC one. But in a QFT, different regularizations differ always by a choice of contact terms and so,

theories defined with different regularizations are related by coupling constant redefinitions. As coupling constants in the worldsheet Lagrangian are precisely the spacetime fields, we conclude that there must be a map between the commutative and the NC fields which maps the ordinary to the NC gauge transformations.

Even before trying to find such a map, it is worth guessing how it will look like. The naive attempt to look for a map among fields and gauge parameters of the form

$$\hat{A} = \hat{A}(A, \partial A, ...; \theta)\,, \qquad \hat{\lambda} = \hat{\lambda}(\lambda, \partial\lambda, ...; \theta)\,, \tag{V.74}$$

would never work. The reason is that this would imply that there is an isomorphism between the ordinary and the NC YM groups; this is impossible since even in the simple U(1) YM case, the one is an abelian group and the other is non-abelian.

The way that Seiberg and Witten managed to find such a map was by relaxing the aim. Instead of looking for a map between gauge groups, one must look for a map between gauge orbits of the groups. Two field configurations related by a gauge transformation must be mapped to two other field configurations related by a gauge transformation. The infinitesimal version of this statement requires then that

$$\hat{A}(A) + \hat{\delta}_{\hat{\lambda}}\hat{A}(A) = \hat{A}(A + \delta_\lambda A)\,. \tag{V.75}$$

Under a modification of (V.74) to include a field-dependent transformation for the gauge parameters $\hat{\lambda} = \hat{\lambda}(\lambda, \partial\lambda, ...; \theta; A)$, Seiberg and Witten managed to solve exactly the equation (V.75) for the $U(1)$ case:

$$\hat{F} = \frac{1}{1 + F\theta} F\,, \tag{V.76}$$

or written in terms of the B-field

$$\hat{F} = B\frac{1}{B + F}F\,. \tag{V.77}$$

This shows that a NC description is not possible in the case that $\mathcal{F} = F + B = 0$ as was to be expected[2].

[2] It would otherwise be really shocking if one could describe a usual commutative YM QFT by means of a NC one. This would mean that the various phenomena that we will encounter later on, like UV/IR mixing, are nothing but an artifact of our description.

V.2.5 Electric Backgrounds

In section V.2.1, when trying to find an $\alpha' \to 0$ limit of string theory, we restricted ourselves to the cases in which $B_{0\mu} = 0$ and postponed the discussion of the electric backgrounds. It is time to justify this separation.

Let us put the electric field in the direction of X^1, so that we take $B_{01} = E \neq 0$. In this subsection, we restrict the range of the indices μ, ν to $\{0, 1\}$ to focus on the electric sector. Writing explicitly for our case the relations (V.45), (V.46) and (V.70) between closed and open string quantities, we find

$$G_{\mu\nu} = G\eta_{\mu\nu}\,, \qquad \theta^{\mu\nu} = \Theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \qquad G_s = g_s\sqrt{1-\tilde{E}^2}\,, \tag{V.78}$$

where we have defined the constants

$$\tilde{E} = \frac{E}{E_{cr}}\,, \qquad E_{cr} = \frac{g}{2\pi\alpha'}\,, \qquad \Theta = \frac{\tilde{E}}{E_{cr}}\frac{1}{1-\tilde{E}^2}\,, \tag{V.79}$$

and G is introduced to later rescale the metric.

We see that the expressions become singular as we increase the value of the electric field until $E \to E_{cr}$. This singularity is indeed related to the fact that the gaussian string vacuum becomes unstable for $E > E_{cr}$. We first give the qualitative picture of what is happening and then the quantitative one. Physically, the strings may be understood as a lace with two oppositely charged particles, one at each endpoint, and both endpoints are forced to lie on the brane. If we turn on an electric field in one of the brane directions, the lace will like to minimize its energy by aligning along the electric field, and each point will pull in opposite directions. The lace does not break because of the tension. However, as the electric field increases, it may happen that the tension is not enough to keep the system stable, and there appears an instability against breaking and creation of new strings. To make it quantitative, one can use the DBI action (which we recall that it is exact for constant field-strengths) in this background. From (II.1) we get

$$\mathcal{L}_{DBI} \sim \sqrt{1-(2\pi\alpha' E)^2}\,, \tag{V.80}$$

which becomes imaginary precisely at $E = E_{cr}$. Yet another way to understand this is by performing a T-duality along the direction of the electric field. Having into account only the (01) coordinates, the resulting configurations describes a relativistic particle moving at speed $v = (E/E_{cr})c$, so that (V.80) becomes

$$\mathcal{L} \sim \sqrt{1-\frac{v^2}{c}}\,. \tag{V.81}$$

The conclusion we draw from this is that we cannot take a large electric field limit as we did in the magnetic case, which required $B \gg 1/\alpha'$. In our present case, however, the electric field must be kept below $1/\alpha'$. It is not hard to convince oneself that there is now way to send $\alpha' \to 0$ keeping finite the open string quantities, from which we must conclude that NC *field* theories with electric θ do not arise as low energy descriptions of string theories.

There is however an interesting limit that can be taken, now that we have a new parameter (E) to play with. The limit is a near-tensionless limit in which $E \to E_{cr}$ and the closed string metric is scaled to infinity. The only way to keep the interactions in this limit (G_s finite) is to put a large number N of D-branes on top of each other, so that effective coupling is actually $G_s^{eff} = G_s N$. One can then let

$$E \to E_{cr}\,, \qquad -g_{00} = g_{11} \sim \frac{1}{1-\tilde{E}^2}\,, \qquad N \sim \frac{1}{\sqrt{1-\tilde{E}^2}}\,, \tag{V.82}$$

and verify that all open string quantities remain finite. It is important to remark that *we are not sending* $\alpha' \to 0$, so that we are not decoupling the massive string states, and *the resulting theory is not a field theory, but a stringy one.*

One of the main properties of the resulting theory is that these open strings do not couple to the closed strings, in apparent contradiction to the general expectation that all open strings contain closed strings. Can't we just bend an open string topology to form a closed one? The answer is that this is not forbidden by any defining property of the theory, but by its dynamics. The intuitive way to understand it is that, being close to its critical value, the electric field keeps that open strings completely rigid, and it takes an infinite energy to bend it to the point of bringing together the endpoints to form a closed string.

Summing up, the limit (V.82) provides us with a new non-critical interacting string theory in which

- open string massive modes still survive,
- the brane becomes invisible to the bulk closed string physics,
- the spacetime seen by the open strings is noncommutative.

This theory has been called NC Open String theory (NCOS)[113, 114].

V.3 Quantum NC Field Theories

Having discussed how NC field theories arise as a low energy description of D-branes in backgrounds with $B_2 \neq 0$, we now move to quantize these theories. The analysis of the quantum properties that follows for the rest of this chapter is purely perturbative. We will see that some of the phenomena we will encounter are rather difficult to understand within our present knowledge of quantum field theories. A non-perturbative analysis is then extra motivated, and this will be the subject of chapter VI.

V.3.1 Perturbative NC ϕ^4

Let us illustrate the main new phenomena that occur in the quantum NC-field theories by using the ϕ^4 theory in 4 dimensions. The action is

$$S = \int dx^4 \left\{ \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi * \phi * \phi * \phi \right\} . \tag{V.83}$$

Note that because of the property (V.33), it is a matter of choice whether to put $*$-products or ordinary ones in the first two terms of the action, and it is obviously easier to leave it as it is.

The equations of motion are obtained as usual by varying the action, and one gets

$$(\Box + m^2)\phi = \frac{\lambda}{6} \phi * \phi * \phi. \tag{V.84}$$

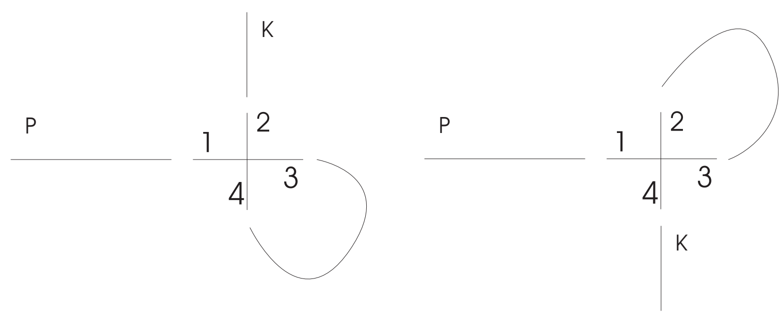
Before going on, let us briefly mention that this equation admits some solutions qualitatively different from its commutative counterpart. In particular, it admits solitonic solutions. This does not violate Derrick's theorem because of the loss of Poincaré invariance due to the presence of the new scale θ.

Let us proceed to obtain the Feynman rules for this theory. Our choice of not including star products in the quadratic part of the action leads to the usual Feynman propagator. We just need to obtain the interaction vertex. In momentum space, we can use the property (V.36) to write

$$\begin{aligned} S_{int} &= \frac{\lambda}{4!} \int d^4x \underbrace{\phi * ... * \phi}_{4} \\ &= \frac{\lambda}{4!} \int \frac{d^4p_1 ... d^4p_4}{(2\pi)^3} \tilde{\phi}(p_1) ... \tilde{\phi}(p_n) \delta^4(p_1 + ... + p_4) \exp(-\frac{i}{2} \sum_{i,j=1,i<j}^{4} p_i \theta p_j). \end{aligned} \tag{V.85}$$

From here we read that the only effect of the noncommutativity is the appearance of the global phase. Momentum is still conserved, but now the vertex is invariant only under cyclic permutations of its 4 legs. An immediate consequence is the classification of diagrams in terms of *planar* and *non-planar* ones, according to whether they can be drawn in a plane or not.

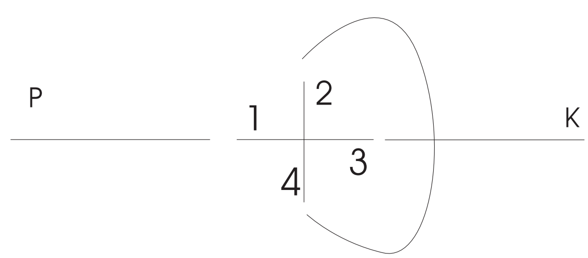
Let us illustrate this in a simple example. Consider a 1-loop diagram in which we have two external legs with 4-momenta p and k. Let us start by attaching the p-leg to one of the four legs of the vertex. Due to cyclic invariance, all choices are equivalent, so we will connect it to leg 1. Now we want to attach the k-leg. If we attach it to leg 2 or to leg 4, we will be able to complete the diagram by pairing the two neighboring legs that remain,



This can be drawn in a plane and it therefore corresponds to a planar diagram. Its phase factor is

$$\exp\left[-\frac{i}{2}p\,\theta k\right], \tag{V.86}$$

which would be trivial if we impose momentum conservation on the external legs, since $p\,\theta p = 0$. Consider now the other option, *i.e.* linking the k-leg to leg 3. This forces us to pair 2 with 4, which cannot be done in a plane, and produces a non-planar diagram,



The important point here is that a phase remains which *depends on the*

internal momentum l

$$\exp\left[-\frac{i}{2}p\,\theta k\right]\exp\left[-i\,l\,\theta k\right]. \tag{V.87}$$

There is a quick way to see that this extra phase will completely change the UV behavior of the diagram. The integration over high l-momentum is now moduled by an infinitely fast oscillating factor which, as we know from distribution theory, tends to the zero distribution and it therefore effectively removes the UV part of the loop.

This example illustrates a rather general fact. It can be shown that all planar diagrams differ from the usual commutative ones by a global phase[3] which depends only on the external momenta. Their divergences are then identical to those in the commutative theory and they do not add any qualitatively new phenomena. Non-planar diagrams, however, are in general self-regulated in the UV by the parameter θ and they require more care. As we will see, they will be responsible for a non-habitual UV/IR mixing.

V.3.2 The 1-loop correction to the self energy and UV/IR mixing

Let us exactly compute the 1-loop correction to the self energy for this NC ϕ^4 theory. *We will only deal with magnetic backgrounds* in this subsection because, as we will see later, NC field theories in electric ones suffer from the problem of lack of unitarity. We will skip most of the explicit calculations here because in section (V.4) we will see in all detail how this works for a similar theory.

As discussed above, the 0th-order contribution is just the usual

$$\Gamma^2_{(0)} = p^2 + m^2\,, \tag{V.88}$$

and the 1st-order one is given by the sum of a planar and a non-planar diagram,

(V.89)

[3] And, of course, a symmetry factor if we compare planar NC diagrams to all commutative ones.

The integrals we need to evaluate are then

$$\Gamma^2_{\mathrm{pl}}(p) = \frac{\lambda}{3} \int \frac{d^4k}{(2\pi)^2} \frac{1}{k^2 + m^2}, \tag{V.90}$$

$$\Gamma^2_{\mathrm{npl}}(p) = \frac{\lambda}{6} \int \frac{d^4k}{(2\pi)^2} \frac{\exp[ip\,\theta k]}{k^2 + m^2}. \tag{V.91}$$

We skip the intermediate steps, but after introducing a UV cutoff Λ, one finds

$$\Gamma^2_{\mathrm{pl}}(p) = \frac{\lambda}{48\pi^2} \left[\Lambda^2 - m^2 \ln\left(\frac{\Lambda^2}{m^2}\right)\right] + \text{finite}, \tag{V.92}$$

$$\Gamma^2_{\mathrm{npl}}(p) = \frac{\lambda}{96\pi^2} m \Lambda_{\mathrm{eff}} K_1\left(\frac{m}{\Lambda^2_{\mathrm{eff}}}\right) = \frac{\lambda}{96\pi^2}\left[\Lambda^2_{\mathrm{eff}} - m^2 \ln\left(\frac{\Lambda^2_{\mathrm{eff}}}{m^2}\right)\right] + \text{finite}, \tag{V.93}$$

where

$$\Lambda^2_{\mathrm{eff}} = \frac{1}{1/\Lambda^2 + p \circ p}\,. \tag{V.94}$$

Due to its often appearance in this type of computations it is worth to define the following $\circ$-product

$$k \circ p \equiv -k^\mu \theta^2_{\mu\nu} p^\nu\,, \tag{V.95}$$

from which it is easy to show that $p \circ p \geq 0$ for magnetic backgrounds.

As promised, the planar diagram is the same as in the commutative theory (this time, even without any external phase) and it has the usual quadratic plus logarithmic divergences in the UV. The non-planar diagram, however, is best expressed in terms of the effective cutoff (V.94). If we just let $\Lambda \to \infty$, then the effective cutoff remains finite

$$\Lambda_{eff} \underset{\Lambda\to\infty}{\longrightarrow} \frac{1}{p \circ p}\,. \tag{V.96}$$

As explained above, this finiteness is due to the fact that the internal phase in (V.91) acts as a regulator.

Notice however that (V.96) is IR divergent after this limit, but it was perfectly IR finite before the limit. One is not even allowed to set θ back to zero after all the UV modes are included! It is clear that the IR physics depend on the exact UV physics of the theory. This is in frontal clash with the Wilsonian picture of renormalization, where field theories should be thought of as coming with an explicit UV cut-off. Renormalization flow is then a flow towards the IR governed by equations that impose that the long

distance physics do not depend on the specific way we used to regularize the theory at the shortest distances.

This point of view seems to be not applicable to NC theories. Despite being causal and unitary (as we will shortly discuss), the physics that we would observe at energy scales of the order of 1 eV are radically different if our theory is given a cut-off at 100 GeV or at 10^{10} GeV! It turns out that precisely the modes between the latter two scales can cause long distance divergences.

One possible reason why this mixing occurs is to interpret the uncertainty relation of NC spacetimes

$$\Delta x \, \Delta y \geq \theta \,, \tag{V.97}$$

as telling us that specifying the theory at shorter and shorter distances in one direction affects more and more the long distance properties in the other direction.

On β-functions:

Here comes an important issue that should be made clear before proceeding: how do we define the β-functions in such theories? Do we consider non-planar diagrams as divergent or as finite? As far as I understand, before discussing β-functions, one should first give a clearer meaning to the renormalization, *i.e.* to the issue of dealing with infinities in these theories (if possible at all!). Nonetheless, the standard rule is to proceed by considering that non-planar diagrams are auto-regulated by their phases[4], so that all divergences come from the planar ones. As their divergent structure coincides with that in commutative theories, the β-functions typically coincide (again, up to symmetry factors discussed in footnote (3)). Let us mention in support of this way of proceeding that the β-functions that we will extract from the string duals of these theories match with the ones obtained in this way.

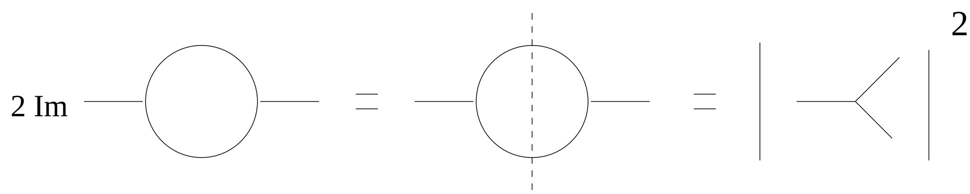
V.3.3 Optical theorem and unitarity

Having established the peculiar behavior of the NC ϕ^4 perturbative expansion, one could wonder whether it is all an artifact of having been dealing with an ill-defined quantum theory. In this section we show that whereas magnetic theories are unitary at 1-loop, electric ones are not. This is in complete agreement with the fact that the magnetic ones arise as the low

[4] In some more complex Feynman diagrams the auto-regulation may require a case-by-case check.

energy description of string theory (and hence inherit unitarity) whereas electric ones do not. Non-unitarity is normally a sign of not having dealt with a complete set of degrees of freedom, which can be interpreted in this case as the string modes that we showed not to decouple in the electric cases.

The optical theorem follows from demanding unitarity of the S-matrix and yields an exact (non-perturbative) requirement. The main power of it is that it provides a non-linear relation from which it follows that the imaginary part of the 1 to 1 forward scattering of a field is equal to the probability of decaying *into an arbitrary final state* (made of an arbitrary number of particles). When the S-matrix is considered in perturbation theory, it yields for a ϕ^3 theory



(V.98)

The first non-trivial condition that follows is at order λ^2. Again we do not give details on this computation since a similar one will be carefully considered in section V.4. We just mention that the result is [27]

$$2\,\mathrm{Im}M = \begin{cases} \frac{\lambda^2}{32\sqrt{p^2}} J_0\left[\frac{\sqrt{1-4m^2/p^2}\sqrt{p^2\, p\circ p}}{2}\right] & , \text{ if } p\circ p > 0\,, \\ \frac{\lambda^2}{32\pi}\int_0^1 dx\, J_0\left[\sqrt{|p\circ p|(m^2+|p^2|x(1-x))}\right] & , \text{ if } p\circ p < 0\,, \end{cases} \tag{V.99}$$

whereas

$$\sum |M|^2 = \begin{cases} \frac{\lambda^2}{32\sqrt{p^2}} J_0\left[\frac{\sqrt{1-4m^2/p^2}\sqrt{p^2\, p\circ p}}{2}\right] & , \text{ if } p\circ p > 0\,, \\ 0 & , \text{ if } p\circ p < 0\,. \end{cases} \tag{V.100}$$

We see that the optical theorem is verified or not depending only on the sign of $p\circ p$ (the $\circ$-product was defined in (V.95). Let us reexamine the correspondence of this sign with magnetic or electric backgrounds.

- For magnetic backgrounds we can box-diagonalize the matrix θ so that its only nonzero entries are $\theta^{23} = -\theta^{32} = \theta$. Then

$$p\circ p = \theta^2\left[(p^2)^2 + (p^3)^2\right] \geq 0\,. \tag{V.101}$$

So $\circ$ provides a definite positive inner product. If any of the components p^2 and p^3 is nonzero, then $p \circ p > 0$ and we are in the first rows of (V.99) and (V.100). The optical theorem is verified at order λ^2. When both momenta are zero, the S-matrix is ill-defined because of the IR singularities mentioned in the previous section.

- For electric backgrounds, we can box-diagonalize the matrix θ so that its only nonzero entries are $\theta^{01} = -\theta^{10} = \theta$. Then

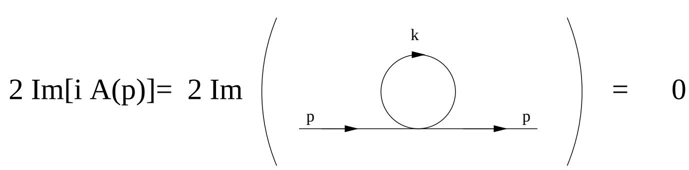
$$p \circ p = \theta^2 \left[(p^0)^2 - (p^1)^2 \right] , \tag{V.102}$$

and the sign becomes negative as soon as $p^0 < |p^1|$. We are then in the second rows of (V.99) and (V.100), so that the optical theorem is not verified. The way to understand the zero appearing in the second row of (V.100) is by noticing that the condition $p^0 > |p^1|$ requires a space-like initial momentum, which makes it kinematically impossible to decay into two massive on-shell particles.

The result is exactly analogous in the NC ϕ^4 theory.

V.3.4 Trying to restore unitarity. The χ-particles.

In the previous section, we illustrated the loss of unitarity in a λ^2 computation for a NC ϕ^3 theory. This is the way this issue was originally discussed in the literature [27]. There is however a much simpler diagram in the massless ϕ^4 theory, which we will use now to study a possible restoration of unitarity. This diagram arises from the condition of order λ imposed by the optical theorem. Since there is no way to produce an order λ contribution in the RHS of the optical theorem (because it is always the square of something), we deduce that the imaginary part of the following diagram should vanish



(V.103)

In purely electric backgrounds, a quick computation yields

$$2\,\mathrm{Im}\left[A(p)\right] = \int \frac{d^4k}{2(2\pi)^3}\,\rho(\lambda,\theta)\,\delta^{(4)}(p-k)\,, \tag{V.104}$$

$$\rho(\lambda,\theta) \sim \frac{\lambda}{\theta^2}\delta\left[(p^0)^2 - (p^1)^2\right] , \tag{V.105}$$

which is clearly nonzero and it therefore shows the loss of unitarity. The point of writing the result in the form (V.104) is that it immediately allows us to reinterpret its RHS as the contribution we would have gotten had we included in the original Lagrangian an extra field χ with coupling $\lambda_{\phi\chi} = \rho^{1/2}(\lambda, \theta)$ to our original field ϕ. Due to the delta function in (V.105), this particle must have the rather strange dispersion relation $k^0 = |k^1|$. Note as well that the coupling diverges in the $\theta \to 0$ limit, a region affected by the discussed UV/IR problems.

However, this interpretation is jeopardized by the fact that if one repeats our analysis for higher powers of λ, the masses of the new required χ-particles are tachyonic. This somehow transforms the problem of perturbative unitarity into an inconsistency of perturbation theory about an unstable vacuum. The conclusion is that although we seem to have a tempting interpretation of the loss of unitarity, and of the possible way of restoring it, it remains almost impossible to verify it quantitatively, at least in perturbation theory.

V.4 Unitarity of non-relativistic NC theories

The purpose of this section is to summarize what we have learnt until now about NC field theories and their quantum properties, and to illustrate it in detail for a particular interesting model. Let us summarize

1. We have explicitly showed that NC ϕ^4 and ϕ^3 field theories are non-unitary when the non-commutativity involves the time coordinate, a case in which they also exhibit an acausal behavior [28]. This is related to the fact that only magnetic theories arise as decoupling limits of string theory.

2. We have also seen that order by order in λ one can try to restore unitarity by adding extra degrees of freedom (χ-particles), although their masses are typically tachyonic. They are proposed to correspond to the instability of the string vacuum in the zero slope limit of electric backgrounds.

These properties will now be reexamined for a *non-relativistic* and NC field theory. It is not straightforward to extend properties 1 and 2 (above) to non-relativistic theories, where the treatment of space and time is completely different, and time non-locality may not lead to the same consequences.

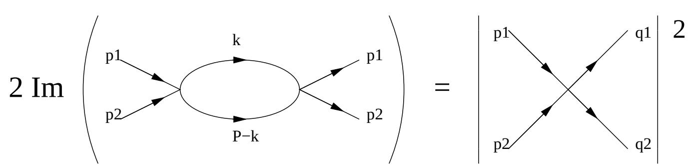
The particular model we will study is a non-relativistic NC ϕ^4 theory in 2+1 dimensions, which can be nicely viewed as the realization of the Galileo group with two central extensions (the mass and the non-commutativity parameter θ)[115]. Our conclusions will be that property 1 is still valid, whereas property 2 fails: there is no way to restore unitarity by the addition of new states even at first non-trivial order in λ. This casts even more doubts on the mentioned interpretation of the χ-particles in the discussed relativistic models.

V.4.1 Four Points Function and Unitarity

To set up our framework, we define a non-relativistic NC scalar field theory in $D = 2+1$ with quartic interactions by the Lagrange density [5], [6]

$$\mathcal{L}_{nr} = \phi^\dagger \left(i\partial_t + \frac{\vec{\nabla}^2}{2} \right) \phi - \frac{\lambda}{4}\, \phi^\dagger * \phi^\dagger * \phi * \phi. \tag{V.106}$$

Following the steps described in section V.3.3, we will study the unitarity of the theory by checking whether the Optical Theorem is fulfilled at the level of two particles scattering. The analog of figure V.98 in our case is then



(V.107)

The left hand side (LHS) and the right hand side (RHS) can be written as

$$\begin{aligned} \text{LHS} \;&\equiv\; 2\,\text{Im}\Bigg(-i\frac{\lambda^2}{2} \cos^2 \frac{\tilde{p}_1 p_2}{2} \\ &\times \int \frac{d^2k dk^0}{(2\pi)^3} \frac{cos^2 \frac{\tilde{P}k}{2}}{[k^0 - \frac{\vec{k}^2}{2} + i\epsilon][p^0 - k^0 - \frac{(\vec{p}-\vec{k})^2}{2} + i\epsilon]} \Bigg), \qquad \text{(V.108)} \\ \text{RHS} \;&\equiv\; \frac{\lambda^2}{4\pi} \cos^2 \frac{\tilde{p}_1 p_2}{2} \int d^3 q_1 \int d^3 q_2\, \delta(q_2^0 - \frac{\vec{q_2}^2}{2}) \delta(q_1^0 - \frac{\vec{q_1}^2}{2}) \\ &\times\; \delta^{(3)}(p_1 + p_2 - q_1 - q_2) \cos^2 \frac{\tilde{q}_1 q_2}{2}\,, \qquad \text{(V.109)} \end{aligned}$$

---

[5] Some perturbative properties of this model in the magnetic case have been studied in [116] and some exact results can be found in [117].

[6] It can be seen that having taken the other possible ordering of the vertex, *i.e.* $\phi^\dagger * \phi * \phi^\dagger * \phi$, would have led to exactly the same unitarity problems.

where $\tilde{p}^\mu \equiv p_\nu \theta^{\nu\mu}$, $P^\mu = p_1^\mu + p_2^\mu$ and the products are defined by $pk \equiv p^0k^0 - \vec{p}\cdot\vec{k}$. Using the identity $2cos^2x = 1 + \cos 2x$ for the cosine involving integrated momenta, both sides can be written as a sum of a planar integral plus a non-planar one. It is straightforward to show that the planar parts are identical in both sides. Therefore, the only job left is to check for the non-planar ones. The RHS can be written as

$$\mathrm{RHS}|_{\mathrm{npl}} = \frac{\lambda^2}{4} cos^2 \tilde{p}_1 p_2 \int \frac{d^3k}{2\pi} \delta(P^0 - k^0 - \frac{(\vec{P}-\vec{k})^2}{2}) \delta(k^0 - \frac{\vec{k}^2}{2}) \cos \tilde{P}k \,, \tag{V.110}$$

irrespective of whether the background is electric of magnetic. The LHS requires considering both cases separately.

V.4.1.1 Magnetic Case

Take the non-commutativity only in the two spatial coordinates $[x,y] = i\theta$. In this case we have $\tilde{P}^0 = 0$ and so we can take the cosine of (V.108) out of the k^0 integral. Therefore, we can perform the k^0 integral using Cauchy's theorem. We are left with

$$\mathrm{LHS}|_{\mathrm{npl}} \;=\; -\frac{\lambda^2}{2(2\pi)^2} \cos^2 \frac{\tilde{p}_1 p_2}{2} \,\mathrm{Im} \int d^2k \frac{cos \tilde{P}k}{P^0 - \frac{\vec{k}^2}{2} - \frac{(\vec{P}-\vec{k})^2}{2} + i\epsilon}. \tag{V.111}$$

The imaginary part is extracted by using that $(x+i\epsilon)^{-1} = \mathcal{P}\frac{1}{x} - i\pi\delta(x)$, and it is then straightforward to show that we obtain exactly (V.110). It can be easily seen that these last two steps are equivalent to replacing the internal propagators by delta functions. Indeed, this is nothing but a proof that the cutting rules are valid for the magnetic case. Notice that we have been able to check the Optical Theorem to all orders in θ.

V.4.1.2 Electric Case

Now, take non-commutativity to affect space and time, *i.e.* $[t,x] = i\theta$. The main difference with respect to the magnetic case is that now $\tilde{P}^0 \neq 0$ and, therefore, the cosine factor in (V.108) cannot be taken out of the k^0 integral. We will find that the order zero θ-term is different from the one we obtain in expanding the RHS (V.110). Furthermore, a linear term arises, in contrast with the RHS, where the first θ term is quadratic. Here one needs to go

through Feynman parameters and residue integrals. We arrive to:

$$\mathrm{LHS}|_{np} = \frac{i\lambda^2}{16\pi}\int_0^1 dx \frac{1}{|1-2x|}\left\{e^{if(P,\theta,x)}\Omega(P^0,\theta)+e^{if(P,-\theta,x)}\Omega(P^0,-\theta)\right\}, \tag{V.112}$$

with

$$\begin{aligned} f(P,\theta,x) &\equiv \frac{|\tilde{P}_0|}{2|1-2x|}\Bigg(2P^0(1-x)-\vec{P}^2x(1-x)) \\ &\quad +\frac{\vec{\tilde{P}}^2}{2|\tilde{P}_0|}|1-2x|-\vec{\tilde{P}}\cdot\vec{P}(1-x)\Bigg), \qquad \text{(V.113)} \\ \Omega(P^0,\theta) &\equiv \Theta(\tilde{P}^0)\Theta(x-\frac{1}{2})+\Theta(\tilde{P}^0)\Theta(\frac{1}{2}-x), \qquad \text{(V.114)} \end{aligned}$$

where we have chosen the symbol $\Theta(x)$ to name the step function, not to be confused with the non-commutative parameter θ.

The integral (V.112) cannot be solved exactly. However, every term in (V.113) is linear in the non-commutativity parameter θ, since $\tilde{P}^0 = \theta P^1$ and $\vec{\tilde{P}} = (\theta P^0, 0)$. Therefore, we can expand the exponentials of (V.112) in order to obtain a power series in θ in the LHS. Some care is needed due to the singular behavior of (V.113) about $x = \frac{1}{2}$, so we will only expand the exponential of the non-singular terms. Taking all this into account we finally obtain

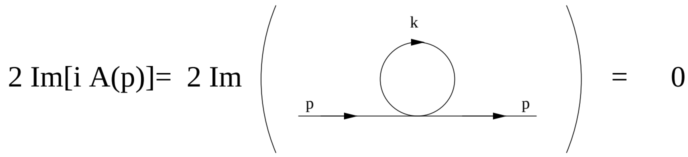
$$\mathrm{LHS}|_{np} = \frac{\lambda^2}{16} + |\theta|\frac{\lambda^2}{32\pi}\left(|P^1|\,\vec{P}^{\,2}+\frac{2(P^0)^2}{|P^1|}\right)+\lambda^2\mathcal{O}(\theta^2). \tag{V.115}$$

The first term arises from expanding a gamma function with imaginary argument, in contrast with the logarithms one finds in the relativistic case [27]. Its value is exactly half of its RHS counterpart (V.110), and so unitarity is violated. The linear term is not present in (V.110) either. Notice that only the absolute value of θ appears in (V.115), in agreement with the original symmetry $\theta \to -\theta$ in (V.108).

V.4.2 Two Points Function and the failure of χ-particles.

Does the method of adding new fine-tuned degrees of freedom to restore unitarity work in this non-relativistic case? Let us try to reproduce the analysis of section V.3.4 and apply the optical theorem to the one-to-one

scattering amplitude, which implies



(V.116)

A short calculation shows that

$$A(p) = -i\lambda \int \frac{dk^0 d^2k}{(2\pi)^3} \frac{cos^2 \frac{\tilde{p}k}{2}}{k^0 - \frac{\vec{k}^2}{2} + i\epsilon} = \frac{-i\lambda}{16\pi}\Lambda^2 \; + \; i\frac{\lambda}{8\pi} \frac{exp\left(\frac{\vec{p}^2}{2\tilde{p}^0}\right)}{|\tilde{p}^0|}. \quad \text{(V.117)}$$

In obtaining this result, we have introduced a hard cutoff for the planar integral (it diverges as in any commutative theory). For magnetic cases, we have $\tilde{p}^0 = 0$. If we take this limit in our result (V.117), we recover the result of [116], *i.e.* $A(p) = \lambda\delta^{(2)}(\vec{p})/4\theta^2$. It has no imaginary part, and so unitarity is preserved. On the contrary, for electric cases $\tilde{p}^0$ is finite, and there is always an imaginary contribution

$$2\, Im\, [A(p)] \; = \; \frac{\lambda}{4\pi} \frac{cos\frac{\vec{p}^2}{2\tilde{p}^0}}{|\tilde{p}^0|}\,, \quad \text{(V.118)}$$

which should be compared to the relativistic formula (V.104). Our result (V.118) can not be interpreted as coming from new particles that couple to our original field, not even if we allow for the coupling to depend on the momenta. This is due to the fact that our expression is a smooth function of the momenta, and so it can never be written as a delta function times a coupling. There is no way to have momentum conservation in the vertices then.

VI. SUPERGRAVITY DUALS OF NONCOMMUTATIVE FIELD THEORIES

In this chapter we link the three major subjects of this thesis: the AdS/CFT duality, its extension to less than maximally supersymmetric cases, and the noncommutative field theories. We develop the technical tools to construct the closed string duals of maximal and less than maximal NC field theories and apply it to obtain the duals of

- a $U(N)$ NC $\mathcal{N}=1$ SYM in 3+1 (section VI.3, reported in [43]),
- a $U(N)$ NC $\mathcal{N}=2$ SYM in 2+1 (section VI.4, reported in [42, 40]).

For the first theory we discuss a good amount of nonperturbative properties derived from the closed string dual: the presence of UV/IR mixing, confinement, the β-function and chiral-symmetry breaking. We will see an interesting property which is absent from the commutative counterpart: the new scale introduced by the noncommutativity can be fine-tuned so that it allows for a decoupling of the KK modes. In constructing the dual of the second theory, we make some precise general remarks about the effect of the 'susy-without-susy' phenomenon in supergravity solutions of NC theories (section VI.4.4) and we analyze its moduli space.

Needless to mention, all cases discussed here involve only magnetic NC theories which, unlike the electric ones, do not suffer from unitarity of causality problems.

VI.1 Introduction

The analysis of NC theories performed until now was purely field-theoretical and perturbative. We saw that one of their most amazing properties is the UV/IR mixing in the non-planar Feynman diagrams of the theory, a property that frontally clashed with the Wilsonian interpretation of renormalization.

The possibility of studying these theories by means of AdS/CFT-like dualities has shed new light on the subject. The supergravity dual is supposed to capture non-perturbative properties of these theories (at least at large N) and renormalization flow has typically the simple interpretation of flowing towards the horizon. The first proposed duals of NC theories are due to Maldacena and Russo [32] and Hashimoto and Itzhaki [33]. In particular, one of these backgrounds is dual to the NC deformation of the usual $\mathcal{N}=4$ SYM in 3+1. We will later discuss how such backgrounds may be constructed and concentrate now on their physical consequences.

- In the first place the NC solution reduces to $AdS_5 \times S^5$ very close to the horizon. By the usual radius/energy relation this implies that in the deep IR the NC field theory reduces to the commutative one. This result is far from trivial as the UV/IR mixing showed that the physics at distances much larger than the NC scale $d \gg \sqrt{\theta}$ did not decouple from those at $d \ll \sqrt{\theta}$. The supergravity result would seem more trustable as it is not based on any perturbative (within supergravity, of course) artifact like Feynman diagrams. They found however that the solution started deviating from $AdS_5 \times S^5$ at scales of order $d \sim R\sqrt{\theta}/l_s$, which in the supergravity approximation ($R \gg l_s$) is much larger than the expected $d \sim \sqrt{\theta}$. The UV/IR mixing could be responsible for modifying the physics until such large distance scales.

- There were some extra difficulties in setting up the computation of Wilson loops, as we will show in a particular case below. They found however that the very deep IR behavior is the same as in the commutative $\mathcal{N}=4$, *i.e.* the energy is proportional to the inverse distance between quarks.

There are two reasons why these results should be taken with care. The first one is that, as we discussed in section V.3.2, the UV/IR mixing is mostly present in the non-planar sector of the theory. We know however that all such diagrams are suppressed against planar ones by factors of N. This means that, as supergravity typically requires $N \gg 1$, we may be dealing with backgrounds which are dual to NC theories collapsed to the planar sector only. This could explain the coincidence in the deep IR between the commutative and the NC supergravity solutions.[1]

[1] See [118, 119] for an extension of the results in [32] to various other maximally supersymmetric NC theories via supergravity, which indicate that commutative and NC field theories may have the same number of degrees of freedom.

The second one is that the $\mathcal{N} = 4$ theory is supposed to be finite as a quantum theory. If this is so, then there is no ambiguity on how to compute even the non-planar diagrams, so there is simply no UV/IR mixing. The earlier than expected deviation from the commutative solution would be then due to a UV/IR mixing at strong coupling which would not be visible in perturbation theory.

This last comment makes it more interesting to extend the duality to non-conformal NC theories which are plagued with UV/IR mixing in almost all observables, and see what supergravity can tell us about it. We will do so in the remaining sections and show that, at least in the case of NC $\mathcal{N} = 1$ SYM in 4d, there seems to be a strong UV/IR mixing that renders the commutative and the NC backgrounds different at all scales.

VI.2 Constructing solutions dual to NC theories with less than 16 supercharges

The aim of this section is to describe the two techniques that have been used to construct supergravity duals of NC field theories with less than 16 supercharges. Essentially we need to find IIA/IIB backgrounds with a nonzero B-field. As we want to study only magnetic noncommutativity, only the space/space components of B are turned on, and they must be constant along the directions of the brane. The requirement that we have less than 16 supercharges leads us to consider D-branes which wrap cycles of special holonomy manifolds.

Note that we do not want to put the B-field along the wrapped directions but along the flat noncompact ones (see figure VI.1). This is because we want to end up with a NC-theory in the flat directions at distances much larger than the cycle. This means that *we do not introduce any flux along the special holonomy manifold*, which implies that all the discussion about covariantly constant spinors, special holonomy, etc., is unchanged. The only effect of the B-field will be a modification of the dilaton and the necessity of turning on another RR-potential. This will be understood when we discuss the second method.

VI.2.1 Method one: brute force

The more pedestrian method consists on making a bosonic ansatz for all the fields that one thinks that should be turned on, then solving the supersymmetry variation of the fermions, and then checking explicitly that

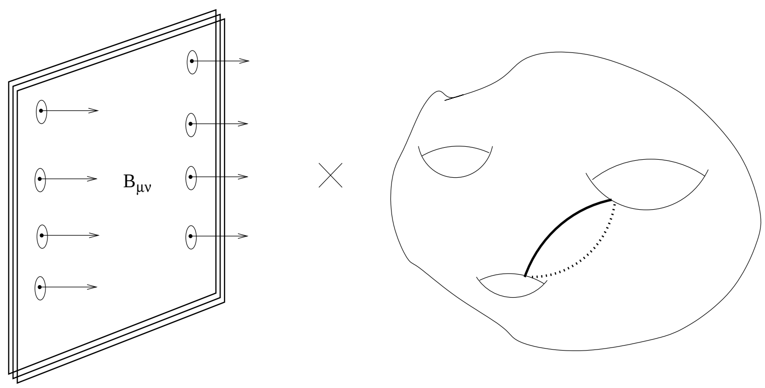


Fig. VI.1: The magnetic B-field is turned on only in the noncompact part of the branes worldvolume, leaving untouched the special holonomy manifold.

they solve the equations of motion. Even if one imposes all the isometries in the ansatz, the process can be 'extremely painful'. It can be however carried out and we will apply it later to construct the NC deformation of the wrapped D6 branes that we discussed in the previous chapter. We were probably successful because the D6 branes have a simple description in 11d supergravity.

Despite being technically complicated, this method has two advantages. First, the construction is very transparent as it is just a matter of making an ansatz for what we look for and solving it. Second, and most important, it necessarily provides the Killing spinors of the background. This can be useful for a number of reasons. First, as we saw in section IV.5.2.3, one can construct most of the covariantly constant tensors in the special holonomy manifold; in particular, the calibrations. Second, it allows one to understand which kind of compactifications will be free from 'supersymmetry without supersymmetry' problems and which ones will not. A remarkable result that we obtained from this method is that, in general, *there is no way to find the supergravity duals of NC theories in the corresponding gauged supergravity.* We will study this in section VI.4.4.

VI.2.2 Method two: T-dualities

The method which finally turns out to be easier to implement exploits the T-duality that is believed to exist between type IIA and IIB compactified on a circle; in particular we will use the fact that a T-duality along a diagonal

direction to a Dp-brane (*i.e.* along a line with nonzero projection along one direction tangent to the brane and one transverse to it) produces Dp-D(p-2) bounds states with a background B-field. This idea was proposed in [120, 121] before the understanding of noncommutative field theories as low-energy limits of string theory, and the technique has been greatly improved in the past two years. Let us briefly review how the original and the improved methods work, and why they are equivalent.

Suppose we have a Dp-brane in flat space along the directions $\{x^0, ..., x^p\}$. We would like to perform a T-duality along a diagonal axis in the plane (x^p, x^{p+1}). Equivalently, we can rotate the brane in that plane and simply T-dualize along x^{p+1}. In the last picture, the originally tilted brane had coordinates satisfying

$$\partial_n \left(x^p + \tan\theta\, x^{p+1}\right) = 0\,, \qquad \partial_t \left(x^p - \cot\theta\, x^{p+1}\right) = 0\,, \tag{VI.1}$$

where ∂_n and ∂_t are normal and tangent derivatives with respect to the string worldsheet's boundary, and θ is the angle of rotation. Now, T-duality along x^{p+1} exchanges Neumann and Dirichlet conditions, so it transforms (VI.1) into

$$\partial_n x^p + \tan\theta\, \partial_t x^{p+1} = 0\,, \qquad \partial_n x^{p+1} - \tan\theta\, \partial_t x^p = 0\,. \tag{VI.2}$$

This mixed boundary conditions can be interpreted as those of a string attached to a $D(p+1)$ brane in the presence of a B-field

$$\partial_n x^\mu \;-\; \mathcal{F}^\mu{}_\nu\, \partial_t x^\nu \;=\; 0\,, \tag{VI.3}$$

where $\mathcal{F}_{[2]} = B_{[2]} + 2\pi\alpha' F_{[2]}$ and, in this case, we have induced $\mathcal{F}_{12} = -\tan\theta$. Such gauge invariant field strength produces D(p-1) charge in the world-volume of the D(p+1) through the Wess-Zumino term.

This is, *grosso modo*, the original method proposed in [120, 121], where it was applied to several cases of branes in flat space to produce various Dp-D(p-2) bound states. What we have seen now is the open string picture of the method, which is a rather simple one. When moving to the closed string picture, the method still had some technicalities that made it difficult to implement in cases other than the description of flat branes in flat backgrounds. Maybe the most relevant difficulty was that T-dualities need to be performed along isometries. In our case, the T-duality was performed along a diagonal direction involving one coordinate along the brane and one transverse to it, and this is not an isometry of the supergravity solution. Originally, this was solved by delocalizing the Dp branes along the x^{p+1} axis before applying the T-duality, for example by adding an infinite

number of parallel branes. In the supergravity solution of flat p-branes, this just amounts to changing slightly the form of the harmonic function $H(r)$: instead of being harmonic in the whole transverse space of dimension $10-p-1$, one can choose it to be harmonic in one dimension less, *i.e.* in a $10-p-2$ space. Schematically,

$$\text{Dp localized}: \qquad H(r) = 1 + \frac{1}{r^{7-p}}\,, \qquad r^2 = \sum_{i=p+1}^{10} (x^i)^2\,. \tag{VI.4}$$

$$\text{Dp delocalized in } \mathrm{x}^{\mathrm{p}+1}: \qquad H(\tilde{r}) = 1 + \frac{1}{\tilde{r}^{6-p}}\,, \qquad \tilde{r}^2 = \sum_{i=p+2}^{10} (x^i)^2\,. \tag{VI.5}$$

As can be seen, delocalizing a brane is fairly simple when we are in flat space and we know the whole geometry solution. The difficulty would increase if we were only given the near horizon region. There, the harmonic function can be very hard to recognize depending on the coordinates we are given. Indeed, if we also abandon flat space backgrounds, the transverse space to the brane is typically a sophisticated fibre bundle, and a better method to delocalize the brane is needed.

The way this can be achieved is just by starting with a brane of one dimension higher, say a D(p+1) along $\{x^0, ..., x^{p+1}\}$ and by T-dualizing along x^{p+1}. In the supergravity dual, one just needs to use the T-duality rules to transform the closed string background. In flat space, it is easy to check that this is equivalent to the replacement (VI.5), no matter if we started with the whole geometry or just the near-horizon.

The last refinement of the original method consists on substituting the rotation of the delocalized brane by a more mechanical algorithm. It just exploits the fact that rotating the brane is equivalent to: first T-dualizing one of the world-volume directions, then turning on a constant B-field, and then T-dualizing back.

Therefore, the improved method for producing the noncommutative configurations can be summarized, from a supergravity point of view, as follows (see also figure VI.2)

(i) Start with a supergravity solution of a Dp along $\{x^0, ..., x^p\}$. We require that at least two of these directions, say $\{x^1, x^2\}$, are flat, while the others may or may not be wrapped along any compact cycle. We compactify x^1 and x^2 on a torus so that ∂_{x^1} and ∂_{x^2} generate circle isometries.

(ii) T-dualize along x^2. This produces a D(p-1) brane delocalized along

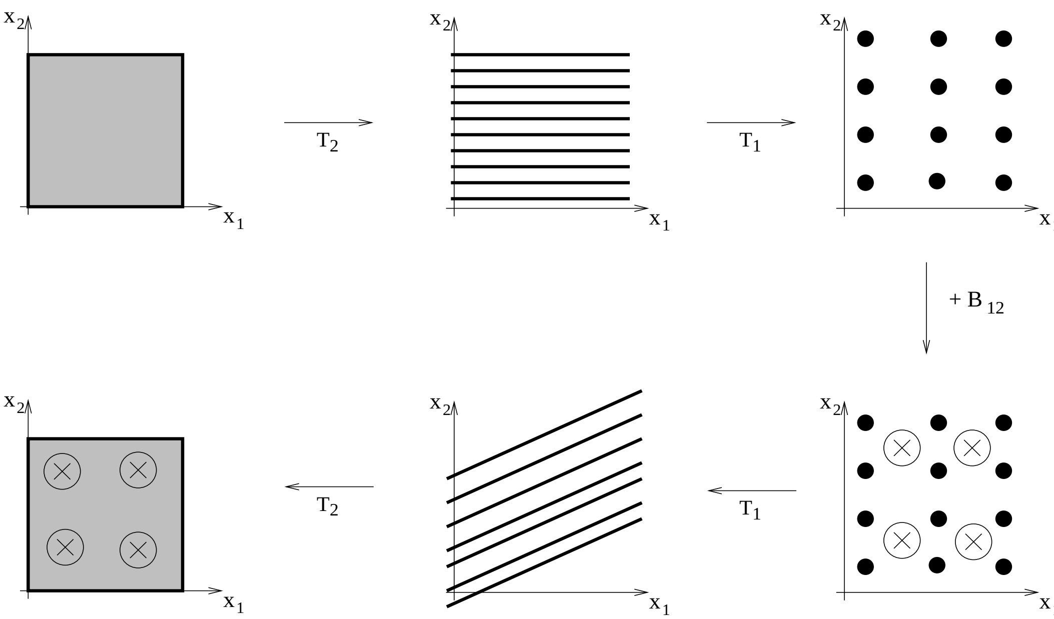


Fig. VI.2: The procedure of consistently introducing a magnetic B-field in a D2-brane. Thick lines are the worldvolume of the branes, circles-with-crosses are magnetic fields and T_i refers to a T-duality along the axis x^i.

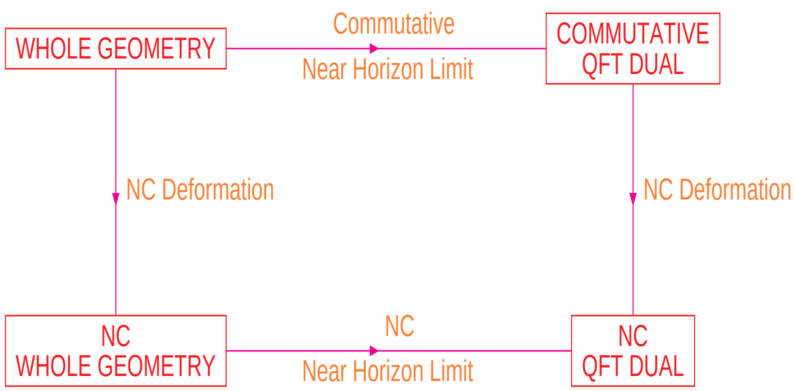
x^2.

(iii) Rotate the D(p-1) along the (x^1, x^2) plane by T-dualizing along x^1, turning on a constant B-field $B = \Theta\, dx^1 \wedge dx^2$, and T-dualizing along x^1 again. The introduction of B-field does not modify the equations of motion because its field strength is zero, and the Chern-Simon's term of the corresponding supergravity Lagrangian is a total derivative.

(iv) T-dualize back on x^2. This is the diagonal T-duality of a delocalized and rotated brane that we mentioned. It produces a bound state of Dp-D(p-2) in the background of a non-trivial B-field. Finally, uncompactify $\{x^1, x^2\}$ by sending the radii of the torus to infinity.

Supersymmetry is preserved throughout this procedure if the spinors originally did not depend on x^1 and x^2 [122], as is typically the case. The introduction of the B-field in step (*iii*) does not break supersymmetry either, since only $H_{[3]} = dB_{[2]} = 0$ appears in the supersymmetry variations of supergravity.

We conclude this section by making a few remarks about the improved method. The first is that it generalizes easily to include B-fields with rank higher than two. The second is the non-trivial fact that, as pointed out in [123], when the B-fields are magnetic, the following diagram holds.



(VI.6)

This is crucial for our purposes, since in the supergravity duals of wrapped branes one only knows the near-horizon region. The third and last remark is that the improved method has been widely used to obtain duals of maximally supersymmetric field theories, as in [32, 33, 124, 125], and of $\mathcal{N}=2$ as in [126].

VI.3 The supergravity dual of the NC $\mathcal{N}=1$ SYM in 3+1

The purpose of this section is to apply the second method discussed above to construct the supergravity dual of the NC version of the 'pure superglue' theory: an $\mathcal{N}=1$ $SU(N)$ SYM in 3+1 without matter supermultiplets. The commutative version has one of the most interesting phenomenologies among all the supersymmetric field theories, mainly because of the possi-

bility of adding chiral matter. This is not possible in theories with more supercharges as the multiplets are too large and left/right matter cannot lie in different susy irreps. The $\mathcal{N}=1$ also enjoys other interesting non-perturbative phenomena like chiral symmetry breaking by fermion bi-linears condensation and confinement.

The outline of what follows in this section is:

- A brief discussion of one of the most successful supergravity duals constructed until now, due to Maldacena and Núñez and dual to the commutative $\mathcal{N}=1$ theory.
- An application of the second method to construct its NC deformation. It will be shown that UV/IR effects seem to persist even in the deep IR, unlike in the dual of a NC $\mathcal{N}=4$. It is also shown that the new scale introduced by the noncommutativity can be used to decouple the KK modes and to a true 3+1 dimensional theory in the IR.
- A computation of the area of the basic Wilson loop in the string theory side. The short distance behavior is different from the commutative theory, but the confining phase is also reached in the IR, with the same string tension as in the commutative theory.
- A computation of the β-function in the supergravity side in both the commutative and the NC versions of this $\mathcal{N}=1$.

VI.3.1 The NC deformation of the Maldacena-Núñez background

We studied in chapter IV how to construct the supergravity duals of field theories with less than maximal supersymmetry. The explicit example we discussed involved D6-branes wrapping Kähler 4-cycles in CY_3 manifolds. In the examples of section IV.4 we discussed from a purely field-theoretical point of view how to perform a twist in a 6d SYM theory in order to put it in $\mathbb{R}^{1,3}\times S^2$. We saw that there were two possible twists, one of them preserving only 4 supercharges. At very low energies, or distances much larger than the S^2, the theory is effectively the $\mathcal{N}=1$ SYM in 4d.

The understanding of the twist allowed Maldacena and Núñez to construct the supergravity solution. Being a 6d theory, the natural branes to look at are D5 branes in type IIB. As their transverse space is $\mathbb{R}\times S^3$, the natural supergravity to construct the solution is 7d gauged supergravity, which appears upon reduction of IIB on S^3. We skip the details here because all the steps are exactly parallel to those of section IV.8.

The final solution represents a stack of N D5 branes wrapping an S^2 inside a Calabi-Yau three-fold, and reads[2]

$$
\begin{aligned}
ds^2_{IIB} &= e^{\Phi}\left[dx^2_{0,3} + N\left(d\rho^2 + e^{2g(\rho)}d\Omega_2 + \frac{1}{4}(w^a - A^a)^2\right)\right], && \text{(VI.7)}\\
F_{[3]} &= dC_{[2]} = \frac{N}{4}\Bigg[-(w^1 - A^1)\wedge(w^2 - A^2)\wedge(w^3 - A^3) \\
&\qquad + \sum_{a=1}^{3} F^a \wedge (w^a - A^a)\Bigg], && \text{(VI.8)}\\
e^{2\Phi} &= e^{2\Phi_0}\,\frac{\sinh 2\rho}{2e^{g(\rho)}}, && \text{(VI.9)}
\end{aligned}
$$

where the definitions of the quantities appearing above are written in the appendix C. To construct the NC deformation we just use the second method explained in the previous section. We skip the intermediate steps and give the result for the case of a magnetic B-field along the $\{x^2, x^3\}$ plane,

$$
\begin{aligned}
ds^2_{IIB} &= e^{\Phi}\left[dx^2_{0,1} + h^{-1}dx^2_{2,3} + N\left(d\rho^2 + e^{2g(\rho)}d\Omega_2 + \frac{1}{4}(w^a - A^a)^2\right)\right], && \text{(VI.10)}\\
F_{[3]} &= dC_{[2]} = \text{unchanged}, && \text{(VI.11)}\\
e^{2\hat{\Phi}} &= e^{2\Phi}h^{-1}, && \text{(VI.12)}\\
B_{[2]} &= -\,\Theta\,\frac{e^{2\Phi}}{h}\,dx^2\wedge dx^3, && \text{(VI.13)}\\
C_{[4]} &= \Theta\,\frac{e^{2\Phi}}{2h}\,C_{[2]}\wedge dx^2\wedge dx^3, && \text{(VI.14)}
\end{aligned}
$$

where we defined

$$h(\rho) = 1 + \Theta^2 e^{2\Phi}. \qquad \text{(VI.15)}$$

Notice that we use $\hat{\Phi}$ for the new value of the dilaton and Φ for the one appearing in (VI.9). Also, Θ is the noncommutative parameter[3], while $C_{[2]}$ and $C_{[4]}$ are the type IIB Ramond-Ramond potentials, with field strengths $F_{[3]}$ and $F_{[5]}$ respectively. Note as well that the B-field is not trivial (its field strength is non-zero) but it is constant along the directions of the brane.

A few remarks concerning (VI.10-VI.15) are in order. First of all, notice that in the commutative limit $\Theta \to 0$ we have $h(\rho) \to 1$ and hence we

[2] We will set $l_s = 1$ in this chapter.

[3] In this chapter we use a capital Θ to denote noncommutativity not to confuse it with the angles of the spheres we will have to deal with.

smoothly recover the whole commutative background of MN. Second, the solution describes a bound state of D5-D3 branes, with the D3 smeared in the world-volume of the D5, and partially wrapped in the two-sphere. If we denote by (θ, ϕ) the coordinates of the S^2 in (VI.10), and by (θ_1, ϕ_1, ψ) the ones of the transverse S^3, we can summarize the configuration in the following array

IIB	x^0	x^1	x^2	x^3	θ	ϕ	ρ	θ_1	ϕ_1	ψ
D5	$-$	$-$	$-$	$-$	$-$	$-$				
D3	$-$	$-$			$-$	$-$				
$B_{[2]}$			$-$	$-$						

Third, as in the MN solution, the metric is completely regular at the origin.

VI.3.1.1 Validity of Supergravity and KK states

Before continuing with our discussion, let us analyze the conditions for the NC-MN solution to be a valid approximation of string theory. The main difference with respect to the commutative solution is that the dilaton does not diverge at the boundary, due to the factor h^{-1} in (VI.11). It acquires its maximum value at infinity -see fig. (VI.3)-, where $e^{\hat{\Phi}} \to \Theta^{-1}$. So if we want to keep small everywhere the corrections coming from higher order diagrams of string theory, we just need to demand

$$\Theta \gg 1 \,. \tag{VI.16}$$

The second validity requirement comes from the curvature. In the non-commutative geometry (VI.10), the scalar of curvature $\mathcal{R}$ vanishes at infinity and it acquires its maximum value at the origin. Requiring the curvature to be small everywhere implies explicitly

$$|\mathcal{R}|_{\max} \,=\, |\mathcal{R}(\rho=0)| \,=\, \frac{32}{3N}\,\frac{e^{-\Phi_0}}{(1+\Theta^2 e^{2\Phi_0})} \,\ll\, 1\,. \tag{VI.17}$$

In order to obtain a truly pure $\mathcal{N}=1$ NC-SYM at low energies, conditions (VI.16) and (VI.17) should be compatible with the decoupling of the massive Kaluza-Klein modes of the wrapped S^2. Since the only change in the metric with respect to the commutative one is in the (x^2, x^3)-plane, the KK modes decoupling condition is exactly the same as in [106], namely

$$N e^{\Phi_0} \ll 1\,. \tag{VI.18}$$

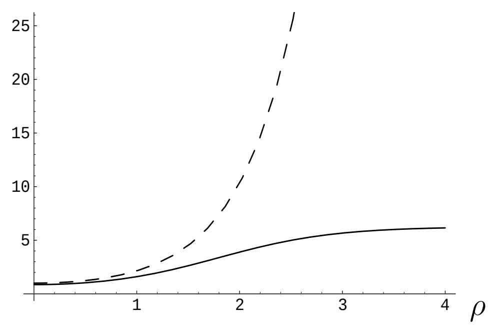


Fig. VI.3: Dilaton behavior as a function of the transverse coordinate. The full line corresponds to $e^{2\hat{\Phi}(\rho)}$ (NC case) and the dashed line to $e^{2\Phi(\rho)}$ (commutative case). While the former remains finite at any value of the variable ρ, the latter blows up at infinity.

It is easy to show that the three inequalities (VI.16)-(VI.18) can be satisfied simultaneously if we choose the three parameters N, Φ_0 and Θ to verify

$$\frac{e^{-3\Phi_0}}{\Theta^2} \ll N \ll e^{-\Phi_0} \ll \Theta\,. \tag{VI.19}$$

We shall see in section 3 that a further restriction will have to be imposed in order to study the quark-antiquark potential. Note that (VI.19) is saying that the price we have to pay to decouple the KK states is to set Θ as the largest length scale of the problem, so we cannot use this to end up with a 'realistic' field theory.

VI.3.1.2 Properties of the solution and UV/IR mixing

As we mentioned already, the NC-MN solution (VI.10) reduces to the commutative one when we send Θ to zero. This corresponds to the fact that, classically, noncommutative theories reduce to commutative ones in this limit. As we saw, this remark does not hold quantum-mechanically and constitutes one of the most interesting facts of the NC field theories, which is related to the so-called UV/IR mixing.

Let us devote our attention to review the metric and the field content of the noncommutative case and carefully analyze if it reduces to its commutative counterpart in the deep IR. Thus we are interested in the $\rho \to 0$ limit of (VI.10). The key observation is that the function $h(\rho)$ tends to the constant value $h(0) = 1 + \Theta^2 e^{2\Phi_0}$. Thus the coefficient multiplying the noncommutative coordinates $dx_2^2 + dx_3^2$ becomes a *constant*, which could have

been absorbed in a rescaling of the coordinates from the very beginning

$$\hat{x}^i = \frac{x^i}{h(0)}\,, \qquad\qquad i=2,3\,. \tag{VI.20}$$

We would like to clarify that although the NC metric seems to tend to the commutative one, *its derivatives do not*. This can be easily inferred for example from the value of the scalar of curvature at $\rho=0$ (VI.17), which does depend on Θ. In general, all objects constructed from derivatives of the metric may differ from their commutative counterparts. The same observation applies to the following analysis for the rest of the fields.

Consider now the B-field, which tends to a constant in this limit, and so it becomes pure gauge. We could have started from the beginning with a gauge-related $\hat{B}$-field

$$\hat{B}_{[2]} = B_{[2]} + d\left(\frac{\Theta}{\Theta^2+e^{-2\Phi_0}}\,x^2\,dx^3\right) \tag{VI.21}$$

that would vanish in the deep IR. In any case, the gauge-invariant field strength $H_{[3]} = dB_{[2]}$ vanishes for $\rho\to 0$. Let us now analyze the dilaton. In this limit, we obtain

$$e^{2\hat{\Phi}} \;\longrightarrow\; \frac{e^{2\Phi_0}}{h(0)} \tag{VI.22}$$

which just amounts to a redefinition of the value of the dilaton at the origin. Furthermore, the field strength $F_{[3]}$ that couples magnetically to the D5 is unchanged everywhere.

Let us now analyze the remaining $C_{[4]}$ that couples to the D3 branes. It is easy to see that in the deep IR limit it does not vanish. Since this is not a gauge-invariant statement, we can look at its field strength[4],

$$F_{[5]} = dC_{[4]} - \frac{1}{2}\left(B_{[2]}\wedge F_{[3]} - C_{[2]}\wedge H_{[3]}\right)\,. \tag{VI.23}$$

Upon substitution we obtain $F_{[5]} = -B_{[2]}\wedge F_{[3]}$, which tends to

$$F_{[5]} \longrightarrow -\frac{\Theta}{e^{-2\Phi_0}+\Theta^2}\; dx^2\wedge dx^3\wedge F_{[3]}\,. \tag{VI.24}$$

when $\rho\to 0$ and, therefore, does not vanish. Indeed, this statement is still coordinate dependent. One way to make it more rigorous is to construct

[4] Indeed, one should make it self-dual by defining $\tilde{F}_{[5]} = \frac{1}{2}(F_{[5]} + *F_{[5]})$ but the following discussion is not affected. Signs are chosen according to the conventions of [120].

scalar quantities out of $F_{[5]}$. One could for example compute $F_{[5]}^2$ where all indices are contracted with the inverse metric. Performing this calculation in the $\mathcal{N}=4$ duals of [32, 33], where the configuration corresponds to D3-D1 bound states instead of D5-D3, one finds that the D1 field strength vanishes quickly in the IR, while the D3 one remains finite. Furthermore, one could compute it as well in the case of D5-D3 in flat space, or more generally in the rest of Dp-D(p-2), directly from [120]. The result is again that the lowest brane field strength vanishes at the origin, while the one of the Dp remains. Nevertheless, in our case, the square of $F_{[5]}$ remains constant too, so that the D3 field strength does not vanish! Therefore, in the deep IR limit, all fields reduce to the commutative result except for the metric and the $F_{[5]}$. For the latter, this difference has its origin in the fact that the MN metric is completely regular at the origin.

Presumably, this could be a signal of the UV/IR mixing that is expected to occur in $\mathcal{N}=1$ and $\mathcal{N}=2$ theories. The observation that in the large N non-planar diagrams are sub-leading with respect to the planar ones, so that noncommutative effects should not be visible, might not apply here because from (VI.19) our solution does not necessarily require to send N to infinity, and it is reasonable to see a different IR behavior from the commutative case.

VI.3.2 Quark-antiquark potential

In this section we obtain the quark-antiquark potential in the $\mathcal{N}=1$ $SU(N)$ field theory by examining the behavior of the Wilson loop. We follow the standard prescription originally given in [127].

The standard way to check if a theory is confining is to introduce an external (non-dynamical) quark-antiquark pair separated a distance L. It is well-known that the potential $V(L)$ between them can be obtained from the expectation value of the Wilson loop

$$W(\mathcal{C}) = \mathrm{Tr}\left[P\exp\left(i\oint_{\mathcal{C}} A\right)\right], \tag{VI.25}$$

by means of the formula

$$\langle W(\mathcal{C})\rangle \sim e^{-T\,V(L)}\,. \tag{VI.26}$$

In these formulae, P denotes the path-ordered integral of the gauge connection A along the contour $\mathcal{C}$ shown in fig. VI.4.

In order to compute the value of (VI.26) in the string theory side, we need to know to which sort of field it couples. To this end, consider pulling

away one brane from a stack of N D-branes. The gauge group is broken $U(N+1) \to U(N) \times U(1)$ and the open strings attached between the stack and the single brane have excitations that correspond to W-bosons, with mass proportional to the separation. The endpoints of these open strings transform in the (anti-)fundamental of $U(N)$, so they look like massive (anti)quarks from the point of view of an observer in the stack. To make these quarks non-dynamical, all we need to do is to pull the brane infinitely far away, so that its mass is higher than any scale we are interested in.

When the stack is replaced by their $AdS_5 \times S^5$ background, the two strings find it energetically favorable to form a bound state (fig. VI.4). These considerations led Maldacena [127] to propose that the Wilson loop (VI.25) acts as a source for open string worldsheet that ends at the boundary of AdS on the contour $\mathcal{C}$. Extending the AdS/CFT dictionary, he proposed that the vev of $W(\mathcal{C})$ can be computed in the string theory side by considering the string partition function in $AdS_5 \times S^5$ with the condition that we have a string worldsheet ending on the loop $\mathcal{C}$. Such a partition function is given in the supergravity approximation simply by the area of the worldsheet with those boundary conditions $A(\mathcal{C})$, so that from (VI.26) we have

$$\langle W(\mathcal{C}) \rangle \sim e^{-T\,V(L)} \sim e^{-A(\mathcal{C})}\,, \qquad \text{(VI.27)}$$

which allows us to straightforwardly compute the $q\bar{q}$-potential.

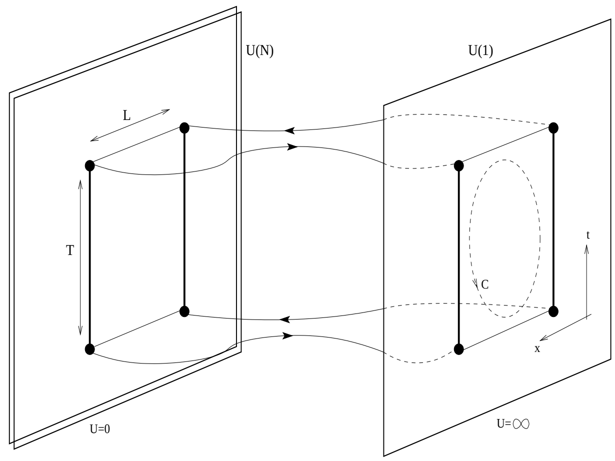
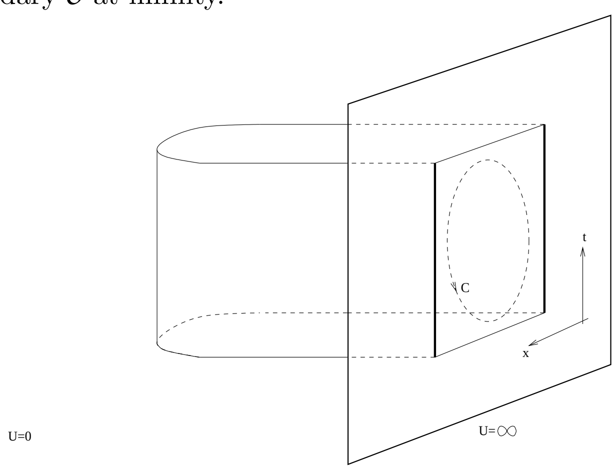


Fig. VI.4: Above, two oppositely oriented strings with one end on the isolated brane and one in the stack. They provide a pair of massive $q\bar{q}$. Below, the stack is replaced by $AdS_5 \times S^5$ and forming a bound state is energetically favored. One is left with a minimal worldsheet with boundary $\mathcal{C}$ at infinity.



VI.3.2.1 Evaluation of the Wilson loop

In the case at hand the Wilson loop average is obtainedby minimizing the Nambu-Goto action in the presence of the $B_{[2]}$ field background

$$S = \frac{1}{2\pi\alpha'}\int d\tau d\sigma \left(\sqrt{-\det \mathrm{g}} + B_{\mu\nu}\partial_\tau X^\mu \partial_\sigma X^\nu\right) , \tag{VI.28}$$

for an open string worldsheet with the mentioned boundary conditions. Explicitly, we want the boundary to define a rectangular loop in the (X^0, X^3)-plane with lengths (T, L). Indeed, if we want to account for the influence of the B-field we need to take a non-static configuration in which the quarks acquire a velocity v in the NC plane. We therefore take the following configuration

$$X_0 = \tau, \quad X_2 = v\tau\,, \quad X_3 = \sigma, \quad \rho = \rho(\sigma)\,, \tag{VI.29}$$

with

$$-L/2 < \sigma < L/2,\;\; 0 < \tau < T\,. \tag{VI.30}$$

Plugging (VI.29) and the NC background (VI.7) in the action, we obtain

$$S = \frac{T}{2\pi}\int_{-\frac{L}{2}}^{\frac{L}{2}} d\sigma \left(H^{-1/2}(\rho)\left(1-\frac{v^2}{h(\rho)}\right)^{1/2}\left(N{\rho'}^2 + \frac{1}{h(\rho)}\right)^{1/2} - \frac{\Theta}{H(\rho)+\Theta^2}v\right) , \tag{VI.31}$$

where $\rho' := \partial_\sigma\rho$ should be understood hereafter and $H(\rho) := e^{-2\Phi}$. In the large T limit, the unrenormalized potential for the $q\bar{q}$ system appears as $S = T\, V_{\mathrm{unren}}$. Notice that in the above expression there are two controllable parameters: the noncommutativity strength Θ and the velocity v of the quarks. From now on, we shall impose the non-supraluminical requirement $|v| < 1$, which ensures $\left(1 - \frac{v^2}{h(\rho)}\right) > 0$.

We can think of the integrand for V_{unren} as a Lagrangian density in classical mechanics with σ as the evolution parameter. Since this Lagrangian density does not depend explicitly on σ, its associated Hamiltonian is a conserved quantity on the extremal of the action:

$$-\frac{1}{h(\rho)H^{-1/2}(\rho)}\left(1-\frac{v^2}{h(\rho)}\right)^{1/2}\left(N{\rho'}^2 + \frac{1}{h(\rho)}\right)^{-1/2} + \frac{\Theta}{H(\rho)+\Theta^2}v \equiv \mathrm{ct}\,. \tag{VI.32}$$

To proceed we evaluate the constant at a special point ρ_0 defined as follows. Locate the boundary of the worldsheet at some distance ρ_{max} from the origin, to be sent to infinity at the end of the calculations. As we increase σ, the worldsheet approaches the origin though the embedding $\rho(\sigma)$ until

it reaches a minimum value ρ_0. By symmetry of the background, this must happen at $\sigma = 0$, so that $\rho_0 = \rho(0)$ and $\rho'(0) = 0$. Evaluating (VI.32) at ρ_0 and solving for ρ' we obtain

$$\rho' = \pm \left(\frac{H(\rho)\,[H(\rho_0) - H(\rho)]}{N} \right)^{1/2} \frac{1}{\alpha^2 + \alpha v\,[\alpha^2 + H(\rho_0)]^{1/2} + H(\rho)}\,, \tag{VI.33}$$

where we have defined the *effective* or boosted noncommutative parameter as

$$\alpha^2 := \frac{\Theta^2}{1 - v^2}\,.$$

Equation (VI.33) can be used to obtain an implicit relation between the quark separation and ρ_0,

$$L(\rho_0) = 2\sqrt{N} \int_{\rho_0}^{\rho_{\max}} d\rho\, \frac{\alpha^2 + \alpha v\,[\alpha^2 + H(\rho_0)]^{1/2} + H(\rho)}{\left(H(\rho)\,[H(\rho_0) - H(\rho)]\right)^{1/2}}\,. \tag{VI.34}$$

Similarly, we can plug equations (VI.32) and (VI.33) into (VI.31) to obtain a relation between the unrenormalized potential and ρ_0,

$$V_{\rm unren}(\rho_0) = \frac{\sqrt{N}}{\pi} \int_{\rho_0}^{\rho_{\max}} d\rho \left(\frac{\Theta^2 + (1 - v^2) H(\rho_0)}{H(\rho)\,[H(\rho_0) - H(\rho)]} \right)^{1/2}\,. \tag{VI.35}$$

Now, from fig. (VI.3), we see that $H(\rho)$ decreases very fast, so that $H(\rho_0) \gg H(\rho)$ for sufficiently large ρ. As a consequence, (VI.35) diverges as we let $\rho_{\max} \to \infty$, which is interpreted as due to the presence of the two bare quark masses at the endpoints of the string. To extract just the potential, we proceed to subtract this contribution as usual [127, 128]. We therefore repeat the calculation for the following configuration

$$X_0 = \tau\,, \quad X_2 = v\tau\,, \quad X_3 \equiv \text{constant}\,, \quad \rho = \sigma\,, \tag{VI.36}$$

which corresponds to a straight worldsheet of a string stretching from the initial stack of N D-branes to the single one located at infinity (see fig. (VI.5.a)). Subtracting this contribution we obtain the following regularized quark-antiquark potential

$$\begin{aligned} V_{\rm ren} \;=\; \frac{\sqrt{N}}{\pi} \Bigg\{ &\int_{\rho_0}^{\rho_{\max}} d\rho \left(\frac{\Theta^2 + (1 - v^2) H(\rho_0)}{H(\rho)\,[H(\rho_0) - H(\rho)]} \right)^{1/2} \\ &- \int_0^{\rho_{\max}} d\rho \left(\frac{(1 - v^2) H(\rho) + \Theta^2}{H(\rho)\,[H(\rho) + \Theta^2]} \right)^{1/2} \Bigg\}\,. \end{aligned} \tag{VI.37}$$

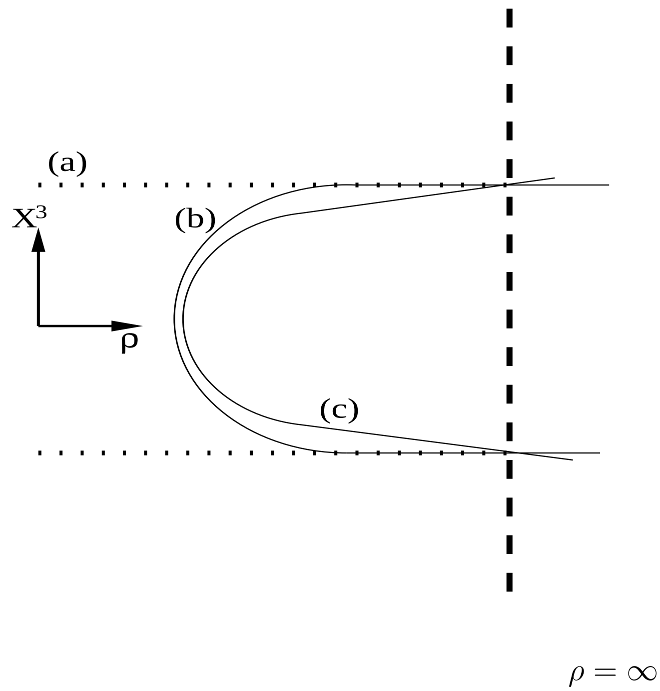


Fig. VI.5: Different configurations for the open string worldsheet in the evaluation of the Wilson loop. (a) corresponds to the subtraction of "two bare quarks". (b) is the only allowed configuration (fine tuned) that leads finite results, for both the potential and the quarks distance. (c) is an example of a configuration that would not cancel the divergences at $\rho \to \infty$. The difference between configurations (b) and (c) is that the first one hits the brane at right angles and, therefore, asymptotes to (a).

It is easy to check that in the commutative limit $\Theta = 0$, both $V_{\rm ren}$ and L remain finite as we let $\rho_{\rm max}$ grow to infinity. Nevertheless, arbitrary values of Θ require a further restriction for the potential to be well-defined. We discuss this issue and its physical interpretation in the next subsection.

VI.3.2.2 The fine tuning

Consider, for a generic value of Θ, the distance between the endpoints of the string in the X^3 axis (VI.34). We want to keep L finite as we move the boundary to $\rho_{\rm max} \to \infty$. Since in this limit $H(\rho) \to 0$, we need

$$\alpha^2 + \alpha v \left[\alpha^2 + H(\rho_0)\right]^{1/2} = 0\,. \qquad \text{(VI.38)}$$

The equation admits two solutions. The first one is $\alpha = 0$, which corresponds to the commutative case, and imposes no restrictions on v. This was to be expected, since in the absence of B-field, Lorentz symmetry is restored in the whole flat part of the brane, and two quarks moving at the same velocity are equivalent to two static quarks. Nevertheless, in the presence of a B_{23}, the Lorentz symmetry is broken, and equation (VI.38) selects

$$v = -\frac{\Theta}{\sqrt{H(\rho_0)}} \, . \tag{VI.39}$$

Since, by equation (VI.34), L determines ρ_0, we see that the velocity must be fine tuned with respect to the strength of the B-field and the distance between quarks. Remarkably, the same fine tuning reappears again when we consider the renormalized potential (VI.37). To obtain a finite potential after the subtraction we need both integrands in (VI.37) to cancel each other when $\rho_{\max} \to \infty$. This imposes the condition

$$\frac{\Theta^2 + (1-v^2)H(\rho_0)}{H(\rho_0)} = 1 \qquad \Rightarrow \qquad v^2 = \frac{\Theta^2}{H(\rho_0)} \, , \tag{VI.40}$$

which is consistent with (VI.39). Therefore, the fine tuning solves simultaneously the problem of fixing the distance between quarks at the boundary at infinity, and the problem of finiteness the potential. Despite being an ad hoc requirement, the fine tuning is necessary to provide a dual supergravity interpretation of the Wilson loop in the field theory.

The physical interpretation is somewhat analogous to the situation when a charged particle enters a region with a constant magnetic field. In that case, there is also a fixed relation -say, a fine tuning- between the three relevant parameters: the radius of the circular orbit, the velocity, and the strength of the magnetic field. As in our case, such a particle would not feel the presence of the magnetic field if it did not have a non-zero velocity transverse to it, which explains why we chose a non-static configuration.

Notice that implementing the fine tuning in (VI.33) shows that now the endpoints of the string hit the boundary at $\rho_{\max} \to \infty$ at right angles, as depicted in fig. (VI.5.b). This is the only way of keeping finite the quarks distance. For instance, the configuration (c) in fig. (VI.5) would not lead to a finite result. In turn, this explains why the fine tuned configuration allows for a finite renormalized potential, since it is the only one that provides an asymptotic coincidence with the configuration that one needs to subtract.

We conclude this subsection by studying the consequences of the requirement that $v < 1$. The fine tuning demands then that $\Theta^2 < H(\rho_0)$.

Since $H(\rho)$ is monotonically decreasing and tends to zero at infinity, this inequality implies two things. The first one is that $H(0) = e^{-2\Phi_0}$ must also satisfy the inequality, so that we need $\Theta^2 < e^{-2\Phi_0}$. This enters in contradiction with the requirements (VI.19) of section VI.3.1.1. Therefore, to properly study the Wilson loop, we have to abandon one of the following requirements: smallness of the dilaton, smallness of the curvature, or KK modes decoupling. If, as in [106], we only disregard the KK condition, we then need to impose

$$1 \ll \Theta < e^{-\Phi_0} \ll N \,. \tag{VI.41}$$

The second one is that ρ_0 has an upper bound ρ_w, for which $H(\rho_w) = \Theta^2$. Choosing $\rho_0 > \rho_w$ would lead to supraluminical velocities [5]. It is easy to see that an upper limit on ρ_0 implies a lower limit on the quark separation L. Seeking for an understanding of this lower limit for L, it is tempting to think that this could be related to the fact that gauge invariant objects in NC theories involve open Wilson lines (see section VI.3.3.2), which exhibit a relation between their lengths and their momentum through

$$\Delta l^\mu = \Theta^{\mu\nu} k_\nu \,. \tag{VI.42}$$

In our case the length L is along X^3 whereas the velocity is along X^2, in agreement with our NC parameter Θ^{23}. A complementary consideration [130] is that relation (VI.42) gives the size of the particle in the X^3 direction when it has a given momentum along X^2. However all these are still vague arguments that we do not claim as conclusive.

VI.3.2.3 The results

Once the necessity for the fine tuning has been discussed, we proceed to apply it to our formulas (VI.34) and (VI.37) to obtain the simplified expressions for the quarks distance and the renormalized potential:

[5] Having [129] in mind, we just mention that ρ_w has the property that the warp factor $e^{\Phi} h^{-1}$ in front of the NC directions of the metric (VI.10) acquires its maximum value.

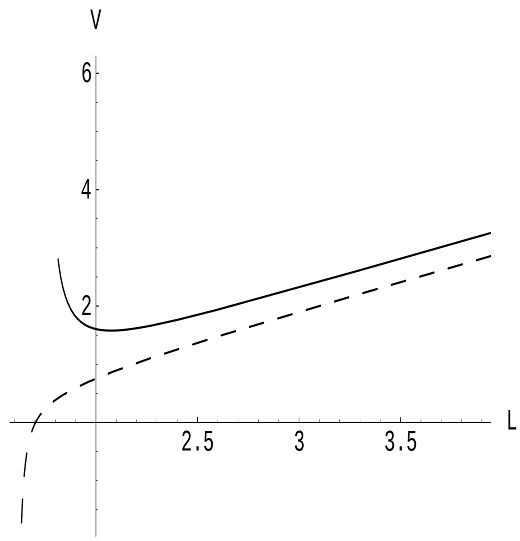


Fig. VI.6: Quark-antiquark potential versus their separation. The dashed line corresponds to the commutative case, while the full curve depicts the corresponding NC one. At large distance, both theories confine, while as we move the quarks closer, the UV physics give a completely different behavior.

$$L = 2\sqrt{N}\int_{\rho_0}^{\infty} d\rho \left(\frac{H(\rho)}{H(\rho_0)-H(\rho)}\right)^{1/2}, \tag{VI.43}$$

$$\begin{aligned} V_{\rm ren} = \frac{\sqrt{N}}{\pi}\Bigg\{ & \int_{\rho_0}^{\infty} d\rho \sqrt{\frac{H(\rho_0)}{H(\rho)\,[H(\rho_0)-H(\rho)]}} \\ & - \int_{0}^{\infty} d\rho \sqrt{\frac{\frac{\Theta^2}{H(\rho_0)}\,[H(\rho_0)-H(\rho)]+H(\rho)}{H(\rho)\,[\Theta^2+H(\rho)]}} \Bigg\}. \end{aligned} \tag{VI.44}$$

Both equations can be used to obtain $V_{\rm ren}$ as a function of L. Although the relation cannot be given algebraically, one can make a numerical plot to study the phases of the theory. In fig.(VI.6), we present the plot of the renormalized potential against the distance between quarks in both the commutative (dashed line) and the NC (full line) backgrounds.

The first immediate observation is that both theories exhibit the same

behavior in the IR. At large separation, the potential is linear in both cases and, restoring α' factors, we obtain

$$V_{\rm ren}(L) \approx \frac{e^{\Phi_0}}{2\pi\alpha'}\, L\,, \tag{VI.45}$$

independent of the value of Θ. Indeed, this result can be proven analytically, and does not rely only on the numerical analysis.

Nevertheless, as we move the quarks closer, the two theories exhibit a very different behavior. In the NC case, the potential becomes extremely repulsive, presumably due to the expected effects of the noncommutative uncertainty relations at short distances. On the other hand, the commutative potential starts deviating from the linear behavior in the opposite way, although this happens in a region where the commutative dilaton (VI.9) is not small anymore, and so the calculation should have been continued in the NS5 S-dual picture.

VI.3.3 Gauge theory physics from noncommutative MN

In this section we try to extract more information of the noncommutative gauge theory from the proposed supergravity dual. Our discussion will be parallel to that in [131, 132], where they studied the commutative Maldacena-Núñez solution. We will follow the conventions of [131]

VI.3.3.1 NC Yang-Mills coupling as a function of ρ

Let us then begin with the discussion on the Yang-Mills coupling for the commutative case. The proposal in [131] is that one can obtain $g_{\rm YM}$ as a function of ρ by the following procedure. Consider the DBI action of a D5 in the background of MN. Take the $\alpha' \to 0$ limit and promote the abelian fields to transform in the adjoint of $SU(N)$. That would give a $SU(N)$ Yang-Mills action in the curved space that the D5 are wrapping, which in our case is $\mathbb{R}^4 \times S^2$. Since we are interested in the IR of the gauge theory, we take a limit in which the volume of the S^2 is small, so that the action becomes, upon an S^2 reduction, a four dimensional $\mathcal{N}=1$ $SU(N)$ SYM with the following bosonic structure

$$S[A_\mu] = -\frac{1}{4g_{\rm YM}^2}\int_{\mathbb{R}^4} d^4x\; F^A_{\alpha\beta}F_A^{\alpha\beta}\,, \qquad\qquad \alpha,\beta = 0,1,2,3\,. \tag{VI.46}$$

Indeed, one would get a series of corrections from the KK modes of the S^2 which, as discussed in section VI.3.1.1, decouple under a certain choice of N,

Θ and Φ_0. The YM coupling appearing in (VI.46) is essentially given by the inverse volume of the S^2 measured with the ten-dimensional commutative metric, and it depends on the radial coordinate ρ.[6]

Let us first adapt this method in order to obtain $\hat{g}_{\rm YM}(\rho)$ for the NC-MN solution (VI.10). We should now expand the DBI including the background B-field. Actually, in the low-energy limit, we have seen that the theory becomes noncommutative. When the dilaton is independent of the gauge theory coordinates, *i.e.* at zero momentum, the DBI action with a constant magnetic B-field gives, in the low-energy limit, the same quadratic terms as its noncommutative version

$$S_{\rm DBI}[\hat{A}_\mu] = \frac{\tau_5}{G_s}\int_{\mathbb{R}^4\times S^2} d^6x\,\sqrt{\det\left(P[G]+2\pi\hat{F}\right)_*} \tag{VI.47}$$

where G_s, $\hat{F}$ and $G_{\mu\nu}$ are the effective coupling constant, field strength and metric seen by the open strings in a B-field background, and τ_5 stands for $(2\pi)^{-5}$. All products in (VI.47) are understood as Moyal $*$-products with noncommutative parameter $\Theta^{\mu\nu}$. We recall that the relations between the open string quantities and the closed string ones $e^{\hat{\Phi}}$, F and $g_{\mu\nu}$ are

$$\begin{aligned} G_{\mu\nu} &= g_{\mu\nu} - (f^*B)_{\mu\rho}g^{\rho\lambda}(f^*B)_{\lambda\nu}\,, \\ \hat{F} &= \frac{1}{1+F\Theta}F\,, \\ G_s &= e^{\Phi}\left(\frac{\det G}{\det g}\right)^{1/4}\,, \\ \Theta^{\mu\nu} &= -g^{\mu\rho}(f^*B)_{\rho\lambda}G^{\lambda\nu}\,. \end{aligned} \tag{VI.48}$$

In order to correctly identify the $\hat{g}_{\rm YM}$ for the noncommutative theory, we use the noncommutative action and variables. Expanding (VI.47) and plugging in our background (VI.10) we obtain

$$S[\hat{A}_\mu] = -\frac{1}{4\hat{g}^2_{\rm YM}}\int_{\mathbb{R}^4} d^4x\;\hat{F}^A_{\alpha\beta} * \hat{F}^{\alpha\beta}_A\,, \qquad \alpha,\beta=0,1,2,3 \tag{VI.49}$$

with the following expression for the noncommutative YM coupling

$$\frac{1}{\hat{g}^2_{\rm YM}(\rho)} = \frac{2\pi^2\tau_5}{N^2G_o}e^{2\Phi(\rho)}e^{-4g(\rho)}\int_{S^2} d\theta d\phi\sqrt{P[G]}\,. \tag{VI.50}$$

[6] In the original paper of Maldacena-Núñez the YM coupling was calculated directly in the gauged supergravity. At the end of the day, it differed from the one in [131] by the fact that the volume of the S^2 was calculated with the seven-dimensional metric. The remarkable matching of [131] with the field theory result seems to select their method.

By explicit calculation, it turns out that the Yang-Mills coupling can be written in the following way

$$\frac{1}{\hat{g}^2_{\mathrm{YM}}(\rho)} = \frac{N}{32\pi^2} Y(\rho) \int_0^{\pi} d\theta \;\sin\theta \left[1 + \frac{\cot^2\theta}{Y(\rho)}\right]^{\frac{1}{2}} \tag{VI.51}$$

where we defined

$$Y(\rho) = 4\rho \coth 2\rho - 1\,. \tag{VI.52}$$

By comparison with [131], we see that the relation between $\hat{g}_{\mathrm{YM}}$ and the radial coordinate turns out to be identical to that of g_{YM}!

VI.3.3.2 Relation between ρ and the energy

To go further and obtain the β-function, we still need to find the relation between ρ and the energy scale of the dual field theory. In the commutative MN, there are two basic lines of argument that lead to the same conclusions. We briefly review them in order to be applied to the noncommutative case.

(i) The authors of [131] observe that $\mathcal{N}=1$ SYM theories have a classical $U(1)_R$ symmetry which is broken at the quantum level (and after considering non-perturbative effects) to Z_2. An order parameter is the vacuum expectation value of the gaugino condensate $<\lambda^2>$, i.e. if $<\lambda^2>\neq 0$, the symmetry is broken. To relate this phenomenon to the supergravity side, one is guided by the fact that we know how the $U(1)_R$ symmetry acts, since it simply corresponds to rotations along the angle ψ. It is easy to realize that such rotations are an isometry of the metric if and only if the supergravity field $a(\rho)$ appearing in (VI.10) is zero[7]. Therefore one is led to conjecture that $a(\rho)$ is the supergravity field dual to the gaugino condensate. The argument finishes by noticing that since $<\lambda^2>$ has protected dimension three, it must happen that [8]

$$<\lambda^2>=\Lambda^3 \tag{VI.53}$$

where Λ is the dynamically generated scale. This leads to the following implicit relation between ρ and the field theory scale μ

$$a(\rho) \propto \frac{\Lambda^3}{\mu^3}\,. \tag{VI.54}$$

[7] Note that both $a(\rho)$ and ψ appear in (VI.10) in an implicit way through the definition of the gauge field A and the left-invatiant forms ω, see (C-3) and (C-5).

[8] The proportionality coefficient is 1 from explicit calculations [133].

(*ii*) A slightly different argument is given in [132]. The authors first expand $g_{\rm YM}(\rho)$ for large ρ (in the UV) where it can be compared to perturbative results of the gauge theory, *i.e.* with $g_{\rm YM}(\mu/\Lambda)$. This immediately gives the searched relation $\rho = \rho(\mu/\Lambda)$, valid in the UV region. Indeed, they also identify $a(\rho)$ as dual to the gaugino condensate by trying to guess what is the exact form of the mass term for the gauginos in the four-dimensional $\mathcal{N} = 1$ SYM. Gauge invariance of the Lagrangian must involve couplings to the gauge field through covariant derivatives. This fact, together with the detailed knowledge of how the twisting of the field theory is performed, allowed the authors to find that the Lagrangian must involve a term like

$$a(\rho)\,\bar{\lambda}\lambda\,. \tag{VI.55}$$

Applying standard arguments of the original AdS/CFT correspondence one would conclude again that $a(\rho)$ is the supergravity field dual to the gaugino condensate.

We now try to adapt these arguments to our NC-MN solution. The first important remark is that noncommutative gauge theories do not have local gauge-invariant operators [134, 135]. Terms like ${\rm Tr}(\hat{F}_{\mu\nu} * \hat{F}^{\mu\nu})$ are only gauge invariant after integration over all the space. This fact increases the difficulty to associate the dual supergravity fields, since they should act as sources of gauge-invariant operators. Nevertheless, since translations are still a symmetry of the theory, there must exist gauge-invariant operators local in momentum space. Such operators involve the so-called open Wilson lines, whose length must be proportional to the transverse momentum. Explicitly, if we name Δl^{μ} the separation between the endpoints of an open Wilson line, and k_{μ} its momentum in the noncommutative directions, gauge-invariance requires

$$\Delta l^{\mu} = \Theta^{\mu\nu} k_{\nu}\,. \tag{VI.56}$$

Several scattering computations [136, 137, 138] seem to confirm that a general supergravity field h couples to the noncommutative version of the ordinary operator to which it coupled when $\Theta = 0$ via

$$\int d^d k\, h(-k)\hat{\mathcal{O}}(k)\,. \tag{VI.57}$$

The noncommutative operator $\hat{\mathcal{O}}(k)$ is defined from its commutative local one $\mathcal{O}(x)$ by inserting the mentioned Wilson line $W[x, \mathcal{C}]$ and Fourier-

transforming[9]

$$\hat{\mathcal{O}}(k) = \mathrm{Tr} P_* \int d^d x \, [W(x,\mathcal{C})\mathcal{O}(y)] * e^{ikx} \tag{VI.58}$$

where $\mathcal{C}$ is a straight path connecting the endpoints separated according to (VI.56) and y is an arbitrary point along $\mathcal{C}$.

The observation is that the relevant fields appearing in our background (VI.10) do not depend on the noncommutative coordinates (x^2, x^3), so that their Fourier-transforms would involve a delta function in momentum space. In other words, we just need the zero-momentum couplings, where the length of the Wilson lines vanishes, and $\hat{\mathcal{O}}$ reduces to $\mathcal{O}$.

We are now ready to apply the arguments (*i*) and (*ii*) to our case. As far as $U(1)_R$ symmetry breaking in the supergravity solution is concerned, nothing changes with respect to the commutative case. Again, shifts of ψ are an isometry of the NC metric if and only if $a(\rho) = 0$. This is due to the fact that the only change in the metric is a factor of $h^{-1}(\rho)$ in front of $dx^2_{2,3}$.

For the same reason, the whole structure of the twisting of the normal bundle to the S^2 inside the Calabi-Yau threefold is also unchanged. At zero-momentum in the noncommutative directions, gauge invariance in the field theory demands again that the fermionic couplings to the gauge fields appear only via covariant derivatives. So it looks like the arguments of (*i*) and (*ii*) lead again to conjecture that $a(\rho)$ is dual to the gaugino condensate.

Indeed, independently of this relation, one could proceed as in [132] and expand $\hat{g}_{\mathrm{YM}}(\rho)$ at very large ρ. In that region, the theory is in the UV and perturbative calculations should be trustable. The field theory results (see the review [139] and references therein) show that the perturbative NC $U(N)$ β-function is identical to the $SU(N)$ commutative one.[10] So both the supergravity behavior of $\hat{g}_{\mathrm{YM}}(\rho)$ and the perturbative behavior of the NC β-function are identical to the commutative case. The conclusion is that the relation between ρ and Λ/μ is also unchanged.

Summarising, it seems like the NC β-function calculated from (VI.10) and the commutative one extracted from the commutative MN are identical. Hence the same results found in [131, 132] hold in our case. We just recall

[9] We refer to *e.g.* [134, 135, 136] for further discussions on the ambiguity of the insertion of $\mathcal{O}(y)$ along the contour $\mathcal{C}$, and for general aspects of open Wilson lines. We shall only make use of a few of their properties.

[10] Recall that the $U(1)$ degrees of freedom inside a noncommutative $U(N)$ gauge theory do not decouple and, unlike the commutative case, they run with the same β-function as the rest of the $SU(N)/Z_N$ [34].

that properly choosing the proportionality function in (VI.54) [140, 141] one remarkably obtains the whole perturbative NSVZ β-function

$$\beta(g_{\rm YM}) = -\frac{3g_{\rm YM}^3 N}{16\pi^2}\left[1-\frac{Ng_{\rm YM}^2}{8\pi^2}\right]^{-1}, \tag{VI.59}$$

and the authors of [131] even identified the contribution of (presumably) non-perturbative fractional instantons.

VI.3.3.3 Phase diagrams

Before finishing, it is worth to restore units and analyze the relevant scales present in the problem. The first comment is that our supergravity solution corresponds already to a near horizon limit. This implies that we have implicitly taken $l_s \to 0$. Restoring units in our background (VI.10) is equivalent to replacing

$$\rho \to l_s\rho\,, \qquad\qquad \Theta \to \frac{\Theta}{l_s^2}\,, \tag{VI.60}$$

with ρ and Θ acquiring now units of energy and energy^{-2} respectively. Notice that there are four dimensionful parameters in the problem, namely, l_s, Λ, Θ and the mass of the KK modes m_{KK}. As mentioned, in units of energy, $m_s = 1/l_s$ is much larger than the rest of scales.

Consider the flow from high-energy to low-energy (see figure VI.7). The analysis of section VI.3.1.1 guarantees that, in the decreasing the energy, the first scale that we find is m_{KK}, which is proportional to the inverse volume of the S^2. As we cross this point, the little string theory on the large S^2 becomes an effective four-dimensional $\mathcal{N}=1$ NC-SYM. The gauge theory is in the perturbative regime as long as we keep the energy much larger than Λ. As we keep decreasing the energy, we approach $\mu \gtrsim \Lambda$ (which happens about $\rho = 0$ according to (VI.54)), perturbation theory breaks down and the theory is best described in terms of our NC supergravity background. Finally, equation (VI.19) tells us that the noncommutative scale $1/\sqrt{\Theta}$ is still at lower energies.

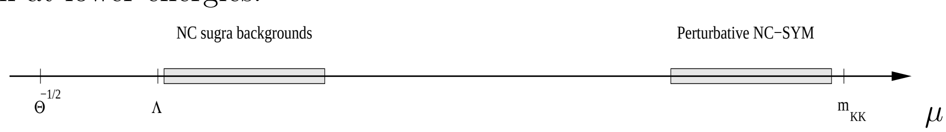


Fig. VI.7: Flow diagram of the theory on the branes.

VI.4 The supergravity dual of a NC $\mathcal{N} = 2$ SYM in 2+1

VI.4.1 Introduction and a little bit of chronology

Let us now switch to the $\mathcal{N} = 2$ SYM in 2+1 that we studied in the previous chapter. Recall that the supergravity solution was found in 8d gauged supergravity and then uplifted to 11d. When going to IIA to study the moduli space of the theory we mentioned that there were two immediate choices of circles to compactify on: one preserved all supersymmetry and the other one none. It is time to justify these statements and to study the impossibility of obtaining the sugra duals of NC theories in gauged supergravities.

To this end, we will construct the NC deformation of the 11d solution (IV.92)-(IV.93) using the first method explained in section VI.2.1, *i.e.* brute force. The ansatz will be performed directly in 11d based on the intuition of how the solution should be. We must say here that the first ansatz we tried was in the 8d gauged supergravity, a technique which had been used in *all* the previous wrapped brane solutions found. After some time playing with the Killing spinor equations we finally proved that there was no supersymmetric solution in 8d supergravity with the isometries that the configuration required!

The approach presented here is different for the sake of clarity. The 11d ansatz will lead us to a set of coupled first order equations by demanding supersymmetry. These will tell us the precise form of the 11d Killing spinors and, from them, we will acquire a better understanding of our 8d problems. At the same time, they will allow us to understand the supersymmetry loss in going to type IIA.

The next section is rather technical. The reader who is not interested in the details may just take a look at the equations (VI.99)-(VI.101) and then jump to the section VI.4.4.

VI.4.2 11d solution of flat NC D6-branes

Before making the ansatz that will lead us to the sugra solution of wrapped D6-branes in the presence of a magnetic B-field, we need to understand how similar solutions look like when the branes are flat in flat space. We will call the latter solutions *NC flat D6-branes* and the former *NC wrapped D6-branes*.

So let us start with the 11d description of the ordinary flat D6 branes (VI.74). Noncommutativity will be put along the (x^5, x^6) plane by introducing a nonzero flux of B_2 along it; this will explicitly break the $SO(1,6)$ isometry of the worldvolume to an $SO(1,4) \times SO(2)$. Uplifting this IIA vocabulary to an 11d one, the ansatz for the metric must be

$$\begin{aligned} ds^2_{(11)} &= \tau^2(r)\left[dx^2_{0,4} + \sigma^2(r)dx^2_{5,6} + H\left(dr^2 + r^2[d\theta^2 + \sin^2\theta d\phi^2]\right)\right] \\ &\quad + \tau^{-4}(r)R^2H^{-1}\left(d\psi + \cos\theta d\phi\right)^2 . \end{aligned} \tag{VI.61}$$

Note that the factor in front of the $U(1)$ fiber is related to the one in front of the other ten coordinates because of the uplifting ansatz (IV.63).We also make an ansatz for the three-form that respects the $U(1)$ monopole fibration

$$A_{[3]} = \chi(r)\ dx^5 \wedge dx^6 \wedge (d\psi + \cos\theta d\phi). \tag{VI.62}$$

We will determine the functions of our ansatz by demanding that the supersymmetry transformations admit a non-trivial Killing spinor. Since the background is bosonic, we just need to care about the gravitino variation

$$\delta\Psi_A = D_A\epsilon - \frac{1}{288}\left(\Gamma_A{}^{BCDE} - 8\delta_A{}^{[B}\Gamma^{CDE]}\right)F_{BCDE}\epsilon, \tag{VI.63}$$

where $D_A = (\partial_A + \frac{1}{4}\omega_A{}^{CD}\Gamma_{CD})$ is the covariant derivative in flat coordinates and F_{BCDE} is the four form field strength.

In what follows it will be very important to make clear the vielbein basis that we are using, since the explicit form of the Killing spinors depends on it. We choose the following vielbein for the diagonal part of (VI.61)

$$\begin{aligned} e^a &= \tau(r)\,dx^a\,, \qquad && a = 0,..,4 \\ e^i &= \tau(r)\sigma(r)\,dx^i\,, && i = 5,6 \\ e^7 &= \tau(r)H^{\frac{1}{2}}(r)\,dr\,, \end{aligned} \tag{VI.64}$$

while for the squashed S^3 we take

$$\begin{aligned} e^8 &= \tau(r)H^{\frac{1}{2}}(r)r\,\tilde{e}^1\,, \\ e^9 &= \tau(r)H^{\frac{1}{2}}(r)r\,\tilde{e}^2\,, \\ e^T &= \tau^{-2}(r)H^{-\frac{1}{2}}(r)R\,\tilde{e}^3\,, \end{aligned} \tag{VI.65}$$

with $\tilde{e}^i$ the usual vielbeins of a round S^3

$$\tilde{e}^1 = d\theta\,, \qquad \tilde{e}^2 = \sin\theta d\phi\,, \qquad \tilde{e}^3 = d\psi + \cos\theta d\phi. \tag{VI.66}$$

Now we proceed to analyze the supersymmetry variations. Due to the $SO(1,4)$ symmetry, the equations for $A=0,1,2,3,4$ are equivalent. If we assume that the Killing spinors do not depend on the coordinates $\{x^0,...,x^6\}$, these equations can be written as

$$(\cos\alpha\Gamma_{D6}+\sin\alpha\Gamma_{D4})\,\epsilon=-\epsilon, \tag{VI.67}$$

with

$$\begin{aligned}\cos\alpha &= \frac{\chi'}{\chi}\tau^3HR^{-1}r^2\,, \qquad \sin\alpha \;=\; -6\tau^3\chi^{-1}\tau'\sigma^2H^{\frac{1}{2}}r^2\,,\\ \Gamma_{D6} &\equiv \Gamma_{0123456}\,, \qquad\qquad\qquad \Gamma_{D4}\equiv\Gamma_{01234T}.\end{aligned} \tag{VI.68}$$

Since $\{\Gamma_{D6},\Gamma_{D4}\}=0$, equation (VI.67) is telling us that we are obtaining a non-threshold bound state of D6-D4 from a IIA point of view, or a bound state MKK-M5 from an M-theory one [142]. To proceed, note that the equation (VI.67) can be rewritten as

$$e^{-\alpha\Gamma_{56T}}\epsilon=-\Gamma_{D6}\epsilon, \tag{VI.69}$$

whose most general solution is

$$\epsilon \;=\; e^{\frac{\alpha}{2}\Gamma_{56T}}\,\tilde\epsilon(r,\theta,\phi,\psi)\,, \qquad \text{with}\;\; \Gamma_{D6}\tilde\epsilon(r,\theta,\phi,\psi)\;=\;-\tilde\epsilon(r,\theta,\phi,\psi). \tag{VI.70}$$

Note that the angle α is a function of r. At this point we need to make an ansatz for $\tilde\epsilon$. Experience suggests

$$\tilde\epsilon(r,\theta,\phi)=f(r)e^{\frac{\theta}{2}\Gamma_{78}}e^{\frac{\phi}{2}\Gamma_{89}}\epsilon_0, \tag{VI.71}$$

where ϵ_0 is a constant spinor verifying $\Gamma_{D6}\,\epsilon_0=-\epsilon_0$. Plugging our ansatz in the remaining supersymmetry variations we obtain the following set of first order, coupled, non-linear BPS equations

$$\begin{aligned}0 &= 3\frac{\tau'}{\tau}+\frac{\sigma'}{\sigma}\,,\\ 0 &= \chi\chi'-6R^2H^{-1}\sigma^4\frac{\tau'}{\tau}\,,\\ 0 &= \frac{3\tau'}{\tau}+\frac{\chi'}{2\chi}+\frac{H'}{2H}\,.\end{aligned} \tag{VI.72}$$

The general solution can be explicitly found and it depends on three arbitrary constants. Two of them can be fixed by demanding that the solution reduces to the commutative one (IV.65) when the $A_{[3]}$ is set to zero (commutative limit). The remaining arbitrary constant has a physical meaning:

it is the strength of the noncommutativity, that we call Θ. The solution is then

$$\begin{aligned} \tau(r) &= h^{\frac{1}{6}}\,, \\ \sigma(r) &= h^{-\frac{1}{2}}\,, \\ \chi(r) &= -\frac{\Theta R}{Hh}\,, \\ f(r) &= h^{\frac{1}{12}}(r)\,, \end{aligned}$$

where $h(r)$ is the equivalent of equation (VI.15) for our case, *i.e.*

$$h(r) = 1 + \Theta^2 H^{-1}\,. \tag{VI.73}$$

Summarizing, the 11d metric, 3-form and the Killing spinors are given by

$$\begin{aligned} ds^2_{(11)} &= h^{\frac{1}{3}}\left(-dx^2_{0,4} + h^{-1}dx^2_{5,6} + H[dr^2 + r^2 d\Omega_2^2]\right)\,, \\ &\quad + H^{-1}h^{-2/3}R\,(d\psi + \cos\theta d\phi)^2 \end{aligned} \tag{VI.74}$$

$$A_{[3]} = -\frac{\Theta R}{Hh}\; dx^5 \wedge dx^6 \wedge (d\psi + \cos\theta d\phi)\,, \tag{VI.75}$$

$$\epsilon(r,\theta,\phi,\psi) = h^{\frac{1}{12}}(r) e^{\frac{\alpha(r)}{2}\Gamma_{56T}} e^{\frac{\theta}{2}\Gamma_{78}} e^{\frac{\phi}{2}\Gamma_{89}} \epsilon_0\,, \tag{VI.76}$$

with the 1/2-preserving projection

$$\Gamma_{D6}\epsilon_0 = -\epsilon_0\,, \tag{VI.77}$$

and the definitions

$$\cos\alpha = h^{-1/2}\,, \qquad \sin\alpha = \Theta(Hh)^{-\frac{1}{2}}\,. \tag{VI.78}$$

This solution describes the whole geometry of N flat NC D6-branes and the number of independent Killing spinors is 16. The configuration corresponds to a bound state of N MKK monopoles and N M5 branes, or a bound state of N D6-D4 branes in type IIA. If we want to use this background *à la* AdS/CFT to study the dual NC field theory, we must take the near horizon limit, which consists of taking $\alpha' \to 0$ keeping fixed

$$u = \frac{r}{\alpha'}\,, \qquad \tilde{\Theta} = \alpha'\Theta\,, \qquad g^2_{YM} = g(\alpha')^{3/2}. \tag{VI.79}$$

After a change of radial variable

$$u = \frac{y^2}{4Ng_{YM}}\,,$$

the metric and the three-form become

$$ds^2_{11} = h^{1/3}\left[dx^2_{0,4} + h^{-1}dx^2_{5,6} + dy^2 + \frac{y^2}{4}\left(d\Omega^2_{(2)} + h^{-1}[d\psi + \cos\theta d\phi]^2\right)\right] \tag{VI.80}$$

$$A_{[3]} = -\frac{\tilde{\Theta}}{4Ng^2_{YM}}\frac{y^2}{h}\, dx^5\wedge dx^6\wedge(d\psi + \cos\theta d\phi)\ , \tag{VI.81}$$

with

$$h(y) = 1 + \left(\frac{\tilde{\Theta}\, y}{2Ng^2_{YM}}\right)^2 . \tag{VI.82}$$

Recall that had we been in the commutative case, this near horizon limit would have yielded the locally flat geometry with an ALE singularity, so we would have found 32 locally preserved supersymmetries; sixteen of them would be killed however by global identifications. In our NC case we do not even find this enhancement at the local analysis that we have just performed.

Let us consider in detail this commutative limit in both the near horizon and the full geometry. Sending $\Theta\to 0$ implies $h\to 1$. In such limit, the full geometry (VI.74) collapses to eq.(IV.65) and the 16 spinors become simply

$$\epsilon(\theta,\phi) = e^{\frac{\theta}{2}\Gamma_{78}}e^{\frac{\phi}{2}\Gamma_{89}}\epsilon_0, \qquad \text{with} \qquad \Gamma_{D6}\epsilon_0 = -\epsilon_0. \tag{VI.83}$$

On the other hand, in the commutative limit, the near horizon region (VI.80) becomes the aforementioned A_{N-1} singularity. Apart from the previous 16 spinors, it also admits the following 16 ones

$$\epsilon(\psi) = e^{-\frac{\psi}{2}\Gamma_{89}}\epsilon_0, \qquad \text{with} \qquad \Gamma_{D6}\epsilon_0 = \epsilon_0. \tag{VI.84}$$

Note that they have a different eigenvalue with respect to Γ_{D6}. Modding out by the Z_N global identifications brings the number of supersymmetries back to 16. Only for $N=1$, flat space, we have a true enhancement of susy.

VI.4.3 11d solution of wrapped NC D6-branes

The analysis perform in order to obtain the NC deformation of the flat D6 background will make it easier to find the corresponding one for D6-branes wrapping a Kähler four-cycle in a CY_3. Again for the sake of simplicity we consider the case when the 4-cycle is an $S^2\times S^2$.

So we reconsider the background we obtained in (IV.92)-(IV.93). Let us turn on a B-field along the (x_1,x_2) plane. As before, we explicitly break

the worldvolume $SO(1,2)$ symmetry to $R \times SO(2)$. As in the unwrapped case, we will also make use of the fact that, in 11d, the factors in front of the 10d part of the metric and in front of the $U(1)$ fiber are related through the lifting ansatz (IV.63). Therefore, our ansatz for the bosonic fields is [11]

$$\begin{aligned} ds^2_{(11)} &= \tau^2(r,\theta)\left[-dx_0^2 + \sigma^2(r,\theta)dx_{1,2}^2 + \frac{3}{2}(r^2+l^2)ds^2_{cycle} + U^{-1}dr^2 \right. \\ &\quad \left. + \frac{r^2}{4}\left(d\theta^2 + mB_{[1]}^2\right)\right] + +\tau^{-4}(r,\theta)\,\tilde{H}^{-1}\,\left[d\phi - Uf^{-1}\cos\theta B_{[1]}\right]^2 , \\ A_{[3]} &= \chi(r,\theta)\; dx^1\wedge dx^2 \wedge \left[d\phi - Uf^{-1}\cos\theta B_{[1]}\right] . \end{aligned} \qquad \text{(VI.85)}$$

Note that we allow the functions of the ansatz to depend on θ. Now we proceed to make an ansatz for the spinor. Just like in the NC flat case, we expect to obtain a projection signaling a bound state of MKK-M5, so we impose

$$\left(\cos\alpha\Gamma_{D6} + \sin\alpha\tilde{\Gamma}_{D4}\right)\epsilon = \epsilon, \qquad \text{(VI.86)}$$

for some angle $\alpha(r,\theta)$ to be determined. Notice that since now the B-field will be along (x^1,x^2), we expect the D4 to span the directions $\{x^0x^3x^4x^5x^6\}$, so that $\tilde{\Gamma}_{D4} = \Gamma_{03456T}$. As in the unwrapped case, see (VI.67) (VI.70), equation (VI.86) implies

$$\epsilon(r,\theta,\phi,\psi) = e^{\frac{\alpha(r,\theta)}{2}\Gamma_{12T}}\tilde{\epsilon}(r,\theta,\phi,\psi)\,, \qquad \text{with} \qquad \Gamma_{D6}\,\tilde{\epsilon} \,=\, \tilde{\epsilon}. \qquad \text{(VI.87)}$$

Now we are ready to obtain the BPS equations by imposing that the supersymmetry variation of the gravitino (VI.63) vanishes. The most immediate relations come from making them compatible for $A=0$ and $A=1,2$ and give [12]

$$3\,\frac{\tau'}{\tau} + \frac{\sigma'}{\sigma} = 0, \qquad\qquad 3\,\frac{\dot{\tau}}{\tau} + \frac{\dot{\sigma}}{\sigma} = 0, \qquad \text{(VI.88)}$$

whose integration yields $\sigma = \tau^{-3}$. The $A=5,6$ equations imply that

$$\left(\partial_\psi + \frac{\Gamma_{36}}{2}\right)\epsilon = 0\,, \qquad \text{and} \qquad \Gamma_{36}\,\epsilon = \Gamma_{45}\,\epsilon\,, \qquad \text{(VI.89)}$$

while the $A=7$ equation implies

$$\tau^{-6} + \tilde{H}\tau^6\chi^2 \,=\, 1\,. \qquad \text{(VI.90)}$$

[11] We use the definitions of (IV.108) for the functions $f(r,\theta)$, $m(r,\theta)$ and $B_{[1]}$.

[12] We use primes for ∂_r and dots for ∂_θ. Also, the integration constant is set to one in order to recover the commutative case when the three-form vanishes.

Now taking a linear combination of the $A = 1,3,9$ equations, and assuming that the spinor does not depend on the fiber coordinate ϕ, one reaches another constraint analogous to (VI.67)

$$(\cos\beta\Gamma_{3689} + \sin\beta\Gamma_{3679})\,\epsilon = -\epsilon, \tag{VI.91}$$

with

$$\cos\beta = U^{\frac{1}{2}} f^{-\frac{1}{2}} \cos\theta, \,, \qquad \sin\beta = f^{-\frac{1}{2}} \sin\theta. \tag{VI.92}$$

Since the matrices Γ_{3689} and Γ_{3679} anticommute, we can proceed as in (VI.70), and rewrite this equation as

$$e^{-\beta\Gamma_{78}}\epsilon = -\Gamma_{3689}\epsilon, \tag{VI.93}$$

whose most general solution is

$$\epsilon(r,\theta,\psi) = e^{\frac{\alpha(r,\theta)}{2}\Gamma_{12T}} e^{\frac{\beta(r,\theta)}{2}\Gamma_{78}} \tilde{\tilde{\epsilon}}(r,\theta,\psi)\,, \qquad \text{with} \qquad \Gamma_{D6}\,\tilde{\tilde{\epsilon}} = -\Gamma_{3689}\,\tilde{\tilde{\epsilon}} = \tilde{\tilde{\epsilon}}. \tag{VI.94}$$

Plugging this into (VI.89) allows us to write down the final ansatz for the spinor:

$$\tilde{\tilde{\epsilon}}(r,\theta,\psi) = \gamma(r,\theta) e^{-\frac{\psi}{2}\Gamma_{89}}\epsilon_0\,, \qquad \text{with} \qquad \Gamma_{D6}\epsilon_0 = -\Gamma_{3689}\,\epsilon_0 = \epsilon_0. \tag{VI.95}$$

The first order BPS equations are

$$\begin{aligned}
0 &= 6\frac{\tau'}{\tau} + \frac{\chi'}{\chi} + \frac{\tilde{H}'}{\tilde{H}}\,, \\
0 &= \dot{\alpha} - \frac{1}{2}\tilde{H}^{\frac{1}{2}}\tau^6\dot{\chi}\,, \\
0 &= \alpha' - \frac{1}{2}\tilde{H}^{\frac{1}{2}}\tau^6\chi'\,, \\
0 &= \frac{\gamma'}{\gamma} - \frac{\tau'}{2\tau}\,, \\
0 &= \frac{\dot{\gamma}}{\gamma} - \frac{\dot{\tau}}{2\tau}\,.
\end{aligned} \tag{VI.96}$$

Luckily, they can be solved analytically and, after fixing the integration constants to reproduce the commutative case when $A_{[3]}$ vanishes, one obtains

$$\begin{aligned}
\tau &= \tilde{h}^{\frac{1}{6}}\,, \\
\chi &= -\frac{\Theta}{\tilde{H}\tilde{h}}\,, \\
\gamma &= \tilde{h}^{\frac{1}{12}}\,, \\
\cos\alpha &= -\tilde{h}^{-\frac{1}{2}}\,, \\
\sin\alpha &= -\Theta(\tilde{H}\tilde{h})^{-\frac{1}{2}}\,,
\end{aligned} \tag{VI.97}$$

with

$$\tilde{h}(r,\theta) = 1 + \Theta^2 \tilde{H}^{-1}(r,\theta). \qquad \text{(VI.98)}$$

So the whole solution for the metric, three-form and Killing spinor is

$$\begin{aligned} ds^2_{(11)} &= \tilde{h}^{\frac{1}{3}} \Big(-dx_0^2 + \tilde{h}^{-1} dx_{1,2}^2 + \frac{3}{2}(r^2+l^2) ds^2_{cycle} + U^{-1} dr^2 \\ &\qquad + \frac{r^2}{4}[d\theta^2 + m B_{[1]}^2] \Big) + \tilde{h}^{-\frac{2}{3}} \tilde{H}^{-1} \left(d\phi - U f^{-1} \cos\theta B_{[1]} \right)^2 , \end{aligned} \qquad \text{(VI.99)}$$

$$A_{[3]} = -\frac{\Theta}{\tilde{H}\tilde{h}}\, dx^1 \wedge dx^2 \wedge \left(d\phi - U f^{-1} \cos\theta B_{[1]} \right) , \qquad \text{(VI.100)}$$

$$\epsilon(r,\theta,\psi) = \tilde{h}^{\frac{1}{12}}(r,\theta) e^{\frac{\alpha(r,\theta)}{2}\Gamma_{12T}} e^{\frac{\beta(r,\theta)}{2}\Gamma_{78}} e^{-\frac{\psi}{2}\Gamma_{89}} \epsilon_0 , \qquad \text{(VI.101)}$$

with the constant spinor ϵ_0 subject to the following 1/8-preserving constraints

$$\Gamma_{D6}\,\epsilon_0 \;=\; \epsilon_0, , \qquad \Gamma_{36}\,\epsilon_0 \;=\; \Gamma_{45}\,\epsilon_0 \;=\; \Gamma_{89}\,\epsilon_0. \qquad \text{(VI.102)}$$

Note that the introduction of the B-field has not broken any extra supersymmetry as expected from the open string picture analysis, so the configuration still preserves 4 real supercharges. This 11d background should be dual in the IR to the 2+1 $\mathcal{N}=2$ $U(\mathrm{N})$ field theory with only a vector multiplet, and with noncommutativity along the (x^1, x^2) plane.

This solution is an M-theory vacuum with fluxes. The topology is $\mathbb{R}^3 \times \mathbb{X}_8$, with $\mathbb{X}_8$ the non Ricci-flat internal manifold. $\mathbb{X}_8$ consists of a complicated four dimensional fibration over the Kähler base space $S^2 \times S^2$. Remarkably, we can smoothly send to zero the noncommutativity, so that the $A_{[3]}$ flux goes to zero and $\mathbb{X}_8$ becomes an $SU(4)$-holonomy Calabi-Yau four-fold. From a IIA perspective it describes a non-threshold bound state of D6-D4 branes with the $D4$ wrapped around the four-cycle. We can describe the configuration by the commonly used arrays as follows,

IIA	x^0	x^1	x^2	θ_1	θ_2	ϕ_2	ϕ_1	r	θ	ψ
D6	–	–	–	–	–	–	–			
D4	–			–	–	–	–			

(VI.103)

11d	x^0	x^1	x^2	θ_1	θ_2	ϕ_2	ϕ_1	r	θ	ψ	ϕ
MKK	–	–	–	–	–	–	–				
M5	–			–	–	–	–				–

(VI.104)

VI.4.4 Susy without susy when going to type IIA and to 8d gauged sugra

Having obtained the Killing spinors in 11d directly, we are in the position to discuss which kind of compactifications will preserve or destroy a fraction of supersymmetry. The method was explained in section IV.9.1 and the following will provide a good set of examples of how the 'susy without susy' phenomenon works. The steps to follow are

1. Put all bosonic fields in a form that fits in the ansatz to reduce. The most important point is that in these type of ansatz *the elementary field is not the metric but the vielbeins*. For example, in going to type IIA, the last vielbein e^T must be chosen such that it does not depend on x^T. On the other hand, when going to 8d, the last three vielbeins must be given in terms of $SU(2)$ invariant 1-forms.

2. The chosen vielbeins provide a base of the tangent space. We must express the Killing spinors in this base and then look at how many of them are left invariant under $U(1)$ or $SU(2)$ transformations when going to IIA or 8d, respectively.[13]

Let us algorithmically apply this procedure case by case.

- **Flat D6-branes.**
  - **From 11d to IIA.** Note that the flat NC D6-branes background (VI.74) has at least two different $U(1)$ isometries, generated by the Killing vectors ∂_ψ and ∂_ϕ. The amount of supersymmetry preserved is different along them. First of all, the vielbeins

[13] This statement can be made more rigorous by computing the more intrinsic Lie-Lorentz derivative [143] with respect to the Killing vectors. For the cases considered in this paper, such derivative collapses to the usual one.

(VI.65) we used in the computation are suitable for a reduction on both circles, as they do not depend on ϕ nor ψ. In this base, the 16 Killing spinors (VI.76) depend on ϕ but not ψ. Thus a reduction along ϕ kills all supersymmetries whereas along ψ they are all preserved. The former leads to the IIA geometry found in *e.g.* [124]. The latter yields an interesting type IIA background as it is solution of the equations of motion which is not supersymmetric,

$$\begin{aligned} ds^2_{IIA} &= \lambda^{\frac{1}{2}} h^{\frac{1}{3}} \left(dx^2_{0,4} + h^{-1} dx^2_{5,6} + H[dr^2 + r^2 d\theta^2] \right) , \\ &\quad + \lambda^{-\frac{1}{2}} h^{-\frac{1}{3}} R r^2 \sin^2\theta \, d\psi^2 , \end{aligned} \tag{VI.105}$$

$$\begin{aligned} e^{4\Phi/3} &= \lambda(r,\theta) , \\ B_{[2]} &= -\frac{\Theta R}{Hh} \cos\theta \, dx^5 \wedge dx^6 , \\ C_{[1]} &= \lambda^{-1} H^{-1} h^{-\frac{2}{3}} R \cos\theta \, d\psi , \\ C_{[3]} &= \frac{\Theta R}{Hh} dx^5 \wedge dx^6 \wedge d\psi , \end{aligned} \tag{VI.106}$$

with

$$\lambda(r,\theta) \equiv H h^{\frac{1}{3}} r^2 \sin^2\theta + H^{-1} h^{-\frac{2}{3}} R \cos^2\theta. \tag{VI.107}$$

There is here a peculiarity. As we mentioned, the near-horizon of the commutative limit of the 11d solution is locally flat space. In this case there are 16 extra Killing spinors preserved (those in (VI.84)). These are only ψ-dependent and survive a ϕ-reduction. Therefore, the $\Theta \to 0$ limit of the IIA background (VI.105)-(VI.106) does preserve 16 supersymmetries.

– **From 11d to 8d.** Here we will encounter a relevant novelty with respect to all other wrapped brane configurations obtained in the literature. The compactification of M-theory on an $SU(2)$ manifold, as it was worked out in [101], requires the use of a vielbein base for the S^3 which is not the one we used in (VI.65). Instead, one has to use the $SU(2)$ invariant one-forms w^i. Our conventions for this section are such that[14]

$$\begin{aligned} w_1 &= -\cos\psi d\theta - \sin\theta \sin\psi d\phi \\ w_2 &= -\sin\psi d\theta + \sin\theta \cos\psi d\phi \\ w_3 &= -d\psi - \cos\theta d\phi. \end{aligned} \tag{VI.108}$$

[14] Note that the signs have been chosen so that both basis share the same orientation.

So instead of (VI.65), one should use

$$\begin{aligned}
\hat{e}^8 &= \tau(r)H^{\frac{1}{2}}(r)r\,w^1 \\
\hat{e}^9 &= \tau(r)H^{\frac{1}{2}}(r)r\,w^2 \\
\hat{e}^T &= \tau^{-2}(r)H^{-\frac{1}{2}}(r)R\,w^3\,.
\end{aligned} \tag{VI.109}$$

We will call (VI.109) the w-base and (VI.65) the e-base. It is easy to work out the form of the spinor in this new base, since we have just performed a local Lorentz transformation which can be shown to consist of a rotation of π along x^9, followed by a rotation of angle $-\psi$ along x^T. The Killing spinors transform with the (inverse) spin $\frac{1}{2}$ representation of such rotations

$$\epsilon' = e^{-\psi\frac{\Gamma_{89}}{2}}e^{\pi\frac{\Gamma_{T8}}{2}}\,\epsilon = \Gamma_{T8}e^{\psi\frac{\Gamma_{89}}{2}}\,\epsilon. \tag{VI.110}$$

Applying this to the Killing spinors (VI.83) we see that they all become $\{\theta,\phi,\psi\}$-dependent in the ω-base, which means that a compactification to 8d supergravity will not preserve a single supersymmetry. Note that all the bosonic fields do fit in the reduction ansatz, so that we still obtain a solution of the 8d gauged sugra equations,

$$\begin{aligned}
ds^2_{(8)} &= \frac{g}{4}y\,h^{1/3}\left(dx^2_{0,4}+h^{-1}dx^2_{5,6}+dy^2\right) \\
e^{\frac{2\phi}{3}} &= \frac{g}{4}y \\
e^{\lambda} &= h^{1/6} \\
G_{[2]} &= -\frac{\Theta g^2}{16Ng^2_{YM}}\frac{y^2}{h}\,dx^5\wedge dx^6 && \text{(VI.111)} \\
G_{[3]} &= -\frac{\Theta g}{4Ng^2_{YM}}\frac{y}{h^2}\,dx^5\wedge dx^6\wedge dy, && \text{(VI.112)}
\end{aligned}$$

where λ is a scalar field on the coset space $\frac{SL(3,R)}{SO(3)}$ and $G_{[2]}$ and $G_{[3]}$ are field strength forms of 8d Sugra.

Its is now understandable what happened when we tried to find the solution in 8d. Despite the fact that the whole solution (VI.111)-(VI.112) fits in the ansatz for the bosonic fields that we were considering, there was no hope to solve it by imposing supersymmetry. Had we worked at the level of the equations of motion, we could have succeeded though.

- **Wrapped D6-branes**

– **From 11d to IIA.** The vielbeins we used in the computations are suitable for reducing along both ϕ and ψ, but the 11d Killing spinors (VI.101) depend only on (θ, ψ). Thus a reduction along ψ destroys all supersymmetry,[15] whereas one along ϕ will produce a type IIA solution preserving the four supercharges. Explicitly,[16]

$$\begin{aligned} ds^2_{IIA} &= e^{2\Phi/3}\tilde{h}^{\frac{1}{3}}\Bigg(-dx_0^2+\tilde{h}^{-1}dx_{1,2}^2+\frac{3}{2}(r^2+l^2)ds^2_{cycle} \\ &\qquad +U^{-1}dr^2+\frac{r^2}{4}[d\theta^2+mB_{[1]}^2]\Bigg)\,, \\ e^{4\Phi/3} &= \tilde{h}^{-\frac{2}{3}}\tilde{H}^{-1}\,, \\ B_{[2]} &= -\frac{\Theta}{\tilde{H}\tilde{h}}\,dx^1\wedge dx^2\,, \\ C_{[1]} &= -Uf^{-1}\cos\theta\, B_{[1]}\,, \\ C_{[3]} &= -\frac{\Theta}{\tilde{H}\tilde{h}}Uf^{-1}\cos\theta\;\; dx^1\wedge dx^2\wedge B_{[1]}. \end{aligned} \qquad \text{(VI.113)}$$

– **From 11d to 8d.** When reducing to 8d gauged sugra, we find a big difference between the commutative and the NC cases, so we analyze them separately. As discussed above, to see amount of supersymmetry preserved in the $SU(2)$ compactification, we have to transform the spinors to the $SU(2)$ left-invariant w-base (VI.108). To do so, we need to apply the rotation (VI.110) to the Killing spinors.

  If we are in the commutative case, it is easy to see that the corresponding spinors become constant, independent of all the S^3 angles. Therefore, the compactification can be performed preserving all four supersymmetries. This is what allowed us to find such solution using 8d supergravity.

  On the other hand, in the NC case, it can be checked that not even the metric can be put in a form that satisfies the reduction ansatz, so the compactification is simply not possible. As a consequence, the NC wrapped D6 solution (VI.99) could have never been found with the usual gauged supergravity method.

[15] The resulting background is, in the commutative limit, the one we used in section IV.9.2 and yielded a zero-dimensional moduli space of the commutative $\mathcal{N}=2$ SYM.

[16] The commutative limit of this background was used in section IV.9.3 and yielded a two-dimensional Kähler moduli space of the commutative $\mathcal{N}=2$ SYM.

VII. HAMILTONIAN FORMALISM FOR NONLOCAL THEORIES

In this section we will develop a Hamiltonian formalism for theories that are non-local in time. Our main concern will be to establish a solid formalism and to apply it to NC theories with electric noncommutativity. Throughout this section, the term *non-local* will always implicitly mean *non-local in time*; spatial non-locality is well understood and its Hamiltonian formalism is straightforward. The whole chapter is based on the papers containing the original construction of the formalism [29, 30] and the results reported in [44].

VII.1 Definition and examples of non-local theories

To simplify the discussion, let us start considering classical mechanics. Standard Lagrangians are functions of $\{q(t), \dot{q}(t), ..., q^{(n)}\}$ with n finite. In other words, they depend on the value of a set of functions *at a given point*, and hence the name of local Lagrangians. The ones we want to deal with here depend on a whole piece of trajectory about a given point t, so that we can write

$$L^{non}(t) = L([q(t+\sigma)])\,, \tag{VII.1}$$

with σ being extendable as far as differentiability of $q(t)$ holds. The best one can do, if Taylor's theorem applies, is to write L^{non} as a function of all time derivatives of q at a given t.

The Euler-Lagrange (EL) equation is obtained by functional variation of (VII.1)

$$\int dt \mathcal{E}(t,t';[q]) = 0\,, \qquad \mathcal{E}(t,t';[q]) \equiv \frac{\delta L^{non}(t)}{\delta q(t')}\,. \tag{VII.2}$$

The main qualitative difference with respect to the standard cases is that the familiar existence and unicity theorems do not apply here. This is because (VII.2) is not a differentiable system. The physical consequence

of this is rather deep. One is (probably) used to giving a set of initial conditions at some initial time and then to interpreting the EL equations as univocally dictating the future of the system. In our case, however, the 'initial conditions' are actually the whole trajectory! Furthermore, not any trajectory is a good 'initial condition' as it may not verify (VII.2). The point of view should then be modified and the dynamics are summarized by saying that (VII.2) defines the hypersurface of allowed trajectories in the space of all possible ones.

Needless to say, equation (VII.2) reduces to the standard EL equations if L^{non} is actually local. In such case we can write

$$L(t) = L(q(t), \dot{q}(t), ..., \overset{(n)}{q}(t)) \quad \Rightarrow \quad \mathcal{E}(t, t'; [q]) = \sum_{m=0}^{n} \frac{\partial L}{\partial \overset{(m)}{q}(t)} \frac{d^m}{dt^m} \delta(t - t')\,,$$

so that (VII.2) yields the familiar equations of motion

$$\sum_{m=0}^{n} \left(-\frac{d^m}{dt^m} \right) \frac{\partial L}{\partial \overset{(m)}{q}(t)} = 0\,. \tag{VII.3}$$

Examples of truly non-local theories are

1. Fokker-Wheeler-Feynman electrodynamics,
2. Regularized local field theories,
3. Some models of meson-nucleon interaction,
4. Semiclassical gravity,
5. String field theory,
6. The p-adic string,
7. Electric NC theories.

Although we will mainly be concerned with the application of our formalism to the last case, it has been recently applied [31] to the study of tachyon condensation in the framework of cases 5 and 6.

Why is it important to have a Hamiltonian formalism?

First of all, the standard method to quantize a theory needs to go through a Hamiltonian treatment; the Hamiltonian functional is promoted to an operator on a Hilbert space and Poisson (or Dirac) parenthesis are promoted to commutators. For nonlocal Lagrangians, there is nothing analog to a Legendre transformation and not even the phase space is well defined. As a bypass, one could take the Lagrangian Path Integral representation as the definition of the quantum field theory

$$Z = \int [d\phi] \epsilon^{-i \int L} \,, \tag{VII.4}$$

but even this expression is normally derived from an Path Integral in phase space and the existence of an hermitian Hamiltonian. One of the properties that will be shown is that, in our formalism, (VII.4) is obtained after integrating out the momenta in a well-defined phase-space Hamiltonian path integral.

VII.2 An equivalent first order Lagrangian

The idea of [29] comes from trying to view the equation (VII.2) still as an 'evolution equation'. Let us imagine that we propose a piece of trajectory $[q(\sigma)]$ as initial condition. If we rewrite (VII.2) as (sometimes this will only be possible implicitly)

$$\ddot{q}_\sigma(t) = \mathcal{F}(t, [q(\sigma)]) \,, \tag{VII.5}$$

where $\mathcal{F}$ is an integro-differential operator, then we can think of it as determining the t-evolution of each point in $q(\sigma)$, which we write as $q_\sigma(t)$. The problem of not having given a proper initial condition $[q(\sigma)]$ is then that it can happen that

$$q_\sigma(t) \neq q(\sigma + t) \,, \tag{VII.6}$$

as we illustrate in figure VII.1.

So, if we think of the EL equation as an equation for a function of *two variables* $\mathcal{Q}(t, \sigma)$, with the boundary condition that at $t = 0$ we recover our proposed solution

$$\mathcal{Q}(0, \sigma) = q(\sigma) \,, \tag{VII.7}$$

then all we need to impose is the constraint that the EL equation must be

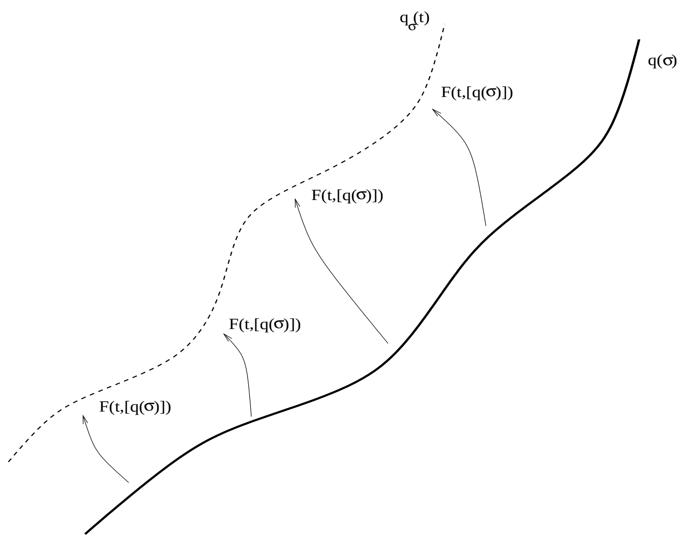


Fig. VII.1: The condition (VII.8) forces these two trajectories to coincide.

compatible with $\mathcal{Q}(t,\sigma)$ being a function only of $t+\sigma$:

$$\frac{\partial}{\partial(t-\sigma)}\,\mathcal{Q}(t,\sigma)=0\,. \tag{VII.8}$$

Note that although we replace $q(t)$ by a function of two variables, the constraint (VII.8) immediately makes it into a function of a single one. Of course this would be just a way of rephrasing the problem if it was not for the great simplifications that this new point of view will bring.

The equivalent Lagrangian $\tilde{L}[\mathcal{Q}(t,\sigma)]$ proposed in [29] is then

$$\tilde{L}[\mathcal{Q}(t,\sigma)]=L^{non}[\mathcal{Q}(t,\sigma)]+\int d\sigma\,\mu(t,\sigma)[\dot{\mathcal{Q}}(t,\sigma)-\mathcal{Q}'(t,\sigma)]\,, \tag{VII.9}$$

where $\dot{}$ and $'$ stand for $\partial/\partial t$ and $\partial/\partial\sigma$ respectively. The field μ is non-dynamical and it just introduces (VII.8) as a primary Lagrangian constraint. This enables us to pass all the original nonlocality in time of $q(t)$ to nonlocality in σ of $\mathcal{Q}(t,\sigma)$ by replacing everywhere in $L^{non}[\mathcal{Q}(t,\sigma)]$

$$q(t+\rho)\rightarrow\mathcal{Q}(t,\sigma+\rho)\,. \tag{VII.10}$$

But then, (VII.9) is actually a first order Lagrangian! This allows for a straight-forward development of a Hamiltonian formalism.

Before going on, we would like to motivate from a more direct point of view the form of the final 1+1 Lagrangian (VII.9). Consider starting from

a non-local Lagrangian, thought of as a function of all the derivatives of q at some point, *i.e.* $L(t) = L(q(t), \dot{q}(t), ..., \overset{(\infty)}{q}(t))$. If we try to apply the usual procedure that makes it into a first order one, we should introduce an infinite set of new variables $\{q_1, q_2, ...\}$ together with an infinite set of constraints forcing

$$q_{n+1} = \frac{d}{dt}\, q_n \,, \qquad n = 1, ..., \infty \,. \tag{VII.11}$$

These can be implemented at the Lagrangian level by introducing an infinite set of Lagrange multipliers μ_n, so that we would end up with a Lagrangian

$$L_{non} = L(q_0, q_1, ..., q_\infty) + \sum_{n=0}^{\infty} \left(\frac{dq_n}{dt} - q_{n+1} \right) \mu_n(t) \,. \tag{VII.12}$$

This is not quite the same as (VII.9) yet, but if we assume that $Q(t, \sigma)$ and $\mu(t, \sigma)$ can be expanded as

$$Q(t, \sigma) = \sum_{n=0}^{\infty} e_n(\sigma) q_n(t) \,, \qquad \mu(t, \sigma) = \sum_{n=0}^{\infty} e_n(\sigma) \mu_n(t) \,, \tag{VII.13}$$

with $e_n(\sigma) = \frac{\sigma^n}{n!}$, then it is immediate to check that the 1+1 Lagrangian (VII.9) reduces to (VII.12). We remark that, however, the 1+1 Lagrangian admits richer dynamics than (VII.12), as it may admit solutions which are not expandable as in (VII.13). It such cases, one must remain in 1+1 to study the system and proceed to its Hamiltonian formalism. We will return to the issue of reducing back to 1 dimension in section VII.3.

Let us go on now with the 1+1 Lagrangian (VII.9). Being first order, it allows for a straight-forward development of a Hamiltonian formalism. In this case one has to take into account the various set of constraints and proceed with the well-known Dirac formalism. We refer the reader to [29] for details and we simply quote here the results, which can be summarized in two steps:

1. The Hamiltonian for a classical mechanics problem of N particles is

$$H(t) \;=\; \int d\sigma \;[\; \mathcal{P}^i(t, \sigma) \mathcal{Q}_i{}'(t, \sigma) \;-\; \delta(\sigma) \mathcal{L}(t, \sigma) \;] \,, \tag{VII.14}$$

with $i = 1, ..., N$ and where the Lagrangian density $\mathcal{L}(t, \sigma)$ is constructed from the original non-local one L^{non} by performing the fol-

lowing replacements

$$\begin{aligned} q_i(t) &\rightarrow \mathcal{Q}_i(t,\sigma)\,, \\ \frac{d^n}{dt^n} q_i(t) &\rightarrow \frac{\partial^n}{\partial\sigma^n}\mathcal{Q}_i(t,\sigma)\,, \\ q_i(t+\rho) &\rightarrow \mathcal{Q}_i(t,\sigma+\rho). \end{aligned} \tag{VII.15}$$

2. There is one Hamiltonian constraint per particle given by[1]

$$\varphi^i(t,\sigma) = \mathcal{P}^i(t,\sigma) \; - \int d\sigma' \; \chi(\sigma,-\sigma') \; \mathcal{E}^i(t;\sigma',\sigma) \;\approx\; 0, \tag{VII.16}$$

where

$$\mathcal{E}^i(t;\sigma',\sigma) = \frac{\delta\mathcal{L}(t,\sigma')}{\delta\mathcal{Q}_i(t,\sigma)}\,, \qquad \chi(\sigma,-\sigma') = \frac{\epsilon(\sigma)-\epsilon(\sigma')}{2}\,, \tag{VII.17}$$

and $\epsilon(\sigma)$ is just the sign distribution.

In principle this is all we need to know to define a Hamiltonian formalism. Everything now follows from

1. The Hamilton equations of motion

$$\dot{\mathcal{Q}}_i(t,\sigma) = \mathcal{Q}_i'(t,\sigma), \tag{VII.18}$$

$$\dot{\mathcal{P}}^i(t,\sigma) = \mathcal{P}^{i\prime}(t,\sigma) + \frac{\delta\mathcal{L}(t,0)}{\delta\mathcal{Q}_i(t,\sigma)} = \mathcal{P}^{i\prime}(t,\sigma) + \mathcal{E}^i(t;0,\sigma), \tag{VII.19}$$

2. The compatibility of these equations of motion with the constraint (VII.16); in other words, from demanding that the evolution dictated by the equations of motion does not move the system away from the hypersurface in phase space determined by the constraints. This leads us to

$$\dot{\varphi}^i(t,\sigma) \;\approx\; \delta(\sigma)\;[\int d\sigma' \; \mathcal{E}^i(t;\sigma',0)] \approx 0. \tag{VII.20}$$

We should require further consistency conditions of this constraint. Repeating this we get an infinite set of Hamiltonian constraints which can be expressed collectively as

$$\tilde{\varphi}^i(t,\sigma) \equiv \int d\sigma' \mathcal{E}^i(t;\sigma',\sigma) \;\approx\; 0, \quad (-\infty<\sigma<\infty). \tag{VII.21}$$

[1] The symbol $\approx$ is used for equations that must hold only on the phase space hypersurface defined by the constraints.

If we use (VII.18) and (VII.15) it reduces to the EL equation (VII.2) of $q_i(t)$ obtained from $L^{non}(t)$.

Summary:

Equation (VII.21) is precisely what we were looking for at the beginning, since now we see that the new 1+1 Hamiltonian system incorporates the EL equation as a constraint on the phase space. It can actually be taken as a proof that our Hamiltonian formalism is equivalent to the original nonlocal system in one dimension less. The advantage is that we are now dealing with a system which is *local in time.* Issues like the construction of the conserved charges, the BRST quantization and the field-antifield formalism follow naturally in the 1+1 formalism.

VII.3 Reducing back the fake non-local theories

One of the self-consistency tests that one must ask to the $d+1$ formalism[2] is that all its phase space quantities reduce back to the ones we would compute in d dimensions when the theory we are dealing with is actually local. We stress that, at the moment, the only consistent formalism in the literature for nonlocal theories is in $d+1$, and that one must remain in $d+1$ to perform any phase space analysis. The reduction to d dimensions is not possible in general, one exception being obviously theories that are actually local. In this section we describe how this reduction works.

Let us then consider a regular higher derivative theory described by the Lagrangian $L(q, \dot{q}, \ddot{q}, ..., q^{(n)})$ and proceed to the $1+1$ formalism as if we were dealing with a non-local Lagrangian. When we embed the higher order theory in the non-local setting, our phase space $T^*J(t) = \{Q(t,\sigma), P(t,\sigma)\}$ becomes infinite dimensional.[3] We expand the phase space quantities in the Taylor basis [144]

$$Q(t,\sigma) = \sum_{m=0}^{\infty} e_m(\sigma)\; q^m(t)\,, \qquad P(t,\sigma) = \sum_{m=0}^{\infty} e^m(\sigma)\; p_m(t), \qquad \text{(VII.22)}$$

[2] We indistinctively use the name $1+1$ or $d+1$ for the formalism with one extra dimension, as it equally applies to mechanics and field theory.

[3] In this subsection we consider only the one particle case for simplicity, so that we remove all subscripts i from the formulas in the previous section.

where $e^\lambda(\sigma)$ and $e_\lambda(\sigma)$ are orthonormal bases

$$e^\lambda(\sigma) \;=\; (-\partial_\sigma)^\lambda \delta(\sigma), \qquad e_\lambda(\sigma) \;=\; \frac{\sigma^\lambda}{\lambda!}. \tag{VII.23}$$

Note that the coefficients in (VII.22) become new canonical variables

$$\{q^m(t),\ p_n(t)\} \;=\; \delta^m{}_n\,. \tag{VII.24}$$

We can now rewrite the Hamiltonian (VII.14) and the two momentum constraints (VII.16) and (VII.21) in this new basis:

$$\begin{aligned}
H(t) &= \sum_{m=0}^{\infty} p_m(t)\, q^{m+1}(t) \;-\; L(q^0, q^1, ..., q^n)\,, && \text{(VII.25)}\\
\varphi_m(t) &= p_m(t) - \sum_{\lambda=0}^{n-m-1} (-D_t)^\sigma\, \frac{\partial L(t)}{\partial q^{\lambda+m+1}(t)} \approx 0\,, && \text{(VII.26)}\\
\psi^m(t) &= (D_t)^m\, [\sum_{\lambda=0}^{n} (-D_t)^\lambda \frac{\partial L(t)}{\partial q^\lambda(t)}] \approx 0\,, && \text{(VII.27)}
\end{aligned}$$

where

$$D_t \;=\; \sum_{r=0} q^{r+1} \frac{\partial}{\partial q^r}. \tag{VII.28}$$

These constraints (VII.26) and (VII.27) are second class and thus they can be used to reduce the dimension of the phase space. It will happen that the reduction is so large that it will turn it into a finite dimensional one, leading to the ordinary Ostrogradski Hamiltonian formalism. The operator D_t defined in (VII.28) will become a time evolution operator for the q's using the first set of the Hamilton equations

$$\dot{q}^r \;=\; q^{r+1}. \tag{VII.29}$$

Using this in (VII.26) the constraints φ_m, $(0 \le m \le n-1)$ coincide with the definition of the Ostrogradski momenta

$$p_m \;\sim\; \sum_{\lambda=0}^{n-m-1} (-\partial_t)^\sigma\, \frac{\partial L(t)}{\partial(\partial_t^{\lambda+m+1} q(t))}\,, \qquad 0 \le m \le n-1. \tag{VII.30}$$

These $n-1$ equations allow to solve for q^λ, $(n \le \lambda \le 2n-1)$ as functions of canonical pairs $\{q^j, p_j\}$, $(0 \le j \le n-1)$,

$$q^\lambda \approx q^\lambda(q^0, q^1, ..., q^{n-1}, p_0, p_1, ..., p_{n-1})\,, \qquad n \le \lambda \le 2n-1. \tag{VII.31}$$

They are combined with the constraints φ_λ, $(n \leq \lambda \leq 2n-1)$

$$\varphi_\sigma = p_\lambda \approx 0\,, \qquad n \leq \lambda \leq 2n-1, \tag{VII.32}$$

to form a second class set and can be used to eliminate the canonical pairs $\{q^\lambda, p_\lambda\}$ $(n \leq \lambda \leq 2n-1)$.

If we take into account (VII.29) the constraint (VII.27) for $m = 0$ becomes the Euler-Lagrange equation for the original higher derivative Lagrangian,

$$\psi^0 \quad \sim \quad \sum_{\lambda=0}^{n} (-\partial_t)^\lambda \frac{\partial L(t)}{\partial(\partial_t^\lambda q(t))} = 0. \tag{VII.33}$$

The constraints (VII.27) for $m > 0$ are the time derivatives of the Euler-Lagrange equation (VII.33) expressed in terms of q's. For a non-singular theory, all such constraints can be rewritten as

$$q^\lambda \quad - \quad q^\lambda(q^0, q^1, ..., q^{n-1}, p_0, p_1, ..., p_{n-1}) \approx 0\,, \qquad \lambda \geq 2n \tag{VII.34}$$

and can be paired with the constraints φ_σ, $(\lambda \geq 2n)$

$$\varphi_\sigma = p_\lambda \approx 0\,, \qquad \lambda \geq 2n\,, \tag{VII.35}$$

forming another set of second class constraints. They can then be used to eliminate the canonical pairs $\{q^\lambda, p_\lambda\}$ $(\lambda \geq 2n)$.

In this way the infinite dimensional phase space is reduced to a finite dimensional one. The reduced phase space is coordinated by $T^*J^n = \{q^l, p_l\}$ with $l = 0, 1, ..., n-1$. All the constraints are second class and the Dirac bracket for these variables has the standard form,

$$\{q^m, p_n\}^* \quad = \quad \delta^m{}_n\,, \qquad \{q^m, q^n\}^* = \{p_m, p_n\}^* = 0. \tag{VII.36}$$

Finally, The Hamiltonian (VII.14) in the reduced space becomes

$$H(t) \quad = \quad \sum_{m=0}^{n-1} p_m(t)\; q^{m+1}(t) \;\; - \;\; L(q^0, q^1, ..., q^n)\,, \tag{VII.37}$$

where the q^n are expressed using (VII.31) as functions of the reduced variables in T^*J^n.

Note that if we consider the limit n going to infinity, the constraints (VII.26) and (VII.27) do not allow, in general, to reduce the dimensionality of the infinite dimensional phase space of the non-local system via Dirac brackets. This shows that the Ostrogradski Hamiltonian formalism does not generalize properly for truly non-local theories.

VII.4 A proper Path Integral quantization

Let us consider the Hamiltonian path integral quantization of the $1+1$ dimensional field theory associated with the Hamiltonian (VII.14) for $L^{\mathrm{non}}(t)$. The path integral in the presence of the two constraints is given by

$$\begin{aligned} \mathcal{Z} &= \int [dP(t,\lambda)][dQ(t,\lambda)]\ \mu \\ &\times \ \exp\left\{ i \int dt d\lambda \left(P(t,\lambda)[\dot{Q}(t,\lambda) - Q'(t,\lambda)] + \tilde{L}(t)\delta(\lambda) \right) \right\}. \end{aligned} \tag{VII.38}$$

The integration is performed over the constrained phase space thanks to the measure [145, 146]

$$\mu = \det \begin{pmatrix} \{\varphi,\varphi\} & \{\varphi,\psi\} \\ \{\psi,\varphi\} & \{\psi,\psi\} \end{pmatrix} \delta(\varphi)\delta(\psi)\,. \tag{VII.39}$$

Using the expansions of the previous section, it is immediate to show that this path integral reduces to the Ostrogradski one in the cases that we deal with a local theory. However, the opposite is not true, in the sense that if we start with the Ostrogradski Path Integral and we just let $n \to \infty$, we recover (VII.38) *without the measure* μ. The system does not have therefore the necessary constraints to be equivalent to the Lagrangian formalism.

We conclude by noting that the Hamiltonian Path Integral (VII.38) reproduces the correct Lagrangian path integral after integrating out the momenta.

VII.5 Hamiltonian symmetry generators

For local theories, symmetry properties of the system are typically examined using the Nöether theorem. In Hamiltonian formalism the relation between symmetries and conservation laws has been discussed extensively for singular[4] Lagrangian systems, for example [147][148]. In this section, we develop a formalism to treat the case of non-local theories.

[4] We recall that a Lagrangian is said to be singular when $\det\left(\frac{\partial L}{\partial \dot{q}\partial \dot{q}}\right) = 0$ which implies that, in trying to build a phase space, not all the momenta can be solved in terms of the velocities.

Suppose we have a non-local Lagrangian like (VII.1) which is invariant under some transformation $\delta q(t)$ up to a total derivative,

$$\delta L^{non}(t) \;=\; \int dt' \, \frac{\delta L^{non}(t)}{\delta q_i(t')} \, \delta q_i(t') \;=\; \frac{d}{dt} k(t). \qquad \text{(VII.40)}$$

Now we move to our $1+1$ dimensional theory and take profit of the fact that it was local in the evolution time t. Therefore, we can construct the corresponding symmetry generator in the Hamiltonian formalism in the usual way

$$G(t) \;=\; \int d\sigma \, [\; \mathcal{P}^i(t,\sigma)\delta\mathcal{Q}_i(t,\sigma) \;-\; \delta(\sigma)\mathcal{K}(t,\sigma) \;], \qquad \text{(VII.41)}$$

where $\delta\mathcal{Q}_i(t,\sigma)$ and $\mathcal{K}(t,\sigma)$ are constructed from $\delta q(t)$ and $k(t)$ respectively by the same replacements (VII.15). The quasi-invariance of the non-local Lagrangian (VII.40), translated to the $1+1$ language, means that

$$\int d\sigma' \, \frac{\delta\mathcal{L}(t,\sigma)}{\delta\mathcal{Q}_i(t,\sigma')} \delta\mathcal{Q}_i(t,\sigma') \;=\; \partial_\sigma \mathcal{K}(t,\sigma). \qquad \text{(VII.42)}$$

When the original non-local Lagrangian has a gauge symmetry the $\delta q_i(t)$ and $k(t)$ contain an arbitrary function of time $\lambda(t)$ and its t derivatives. In $\delta\mathcal{Q}_i(t,\sigma)$ and $\mathcal{K}(t,\sigma)$ the $\lambda(t)$ is replaced by $\Lambda(t,\sigma)$ in the same manner as $q_i(t)$ is replaced by $\mathcal{Q}_i(t,\sigma)$ in (VII.15). However in order for the transformation generated by (VII.41) to be a symmetry of the Hamilton equations, $\Lambda(t,\sigma)$ can not be an arbitrary function of t but it should satisfy

$$\dot{\Lambda}(t,\sigma) \;=\; \Lambda'(t,\sigma) \qquad \text{(VII.43)}$$

as will be shown shortly. This restriction on the parameter function Λ means that the transformations generated by $G(t)$ in the $d{+}1$ dimensional Hamiltonian formalism are *rigid* transformations in contrast to the original ones for the non-local theory which are *gauge* transformations. In the appendix D we will see how this rigid transformations in the $d{+}1$ dimensional Hamiltonian formalism are reduced to the usual gauge transformations in d dimension for the $U(1)$ Maxwell theory.

The generator $G(t)$ generates the transformation of $\mathcal{Q}_i(t,\sigma)$,

$$\delta\mathcal{Q}_i(t,\sigma) \;=\; \{\mathcal{Q}_i(t,\sigma),\; G(t)\}, \qquad \text{(VII.44)}$$

corresponding to the transformation $\delta q_i(t)$ in the non-local Lagrangian. It also generates the transformation of the momentum $\mathcal{P}^i(t,\sigma)$ and so, of any

functional of the phase space variables. In particular, we will see that, as consistency demands, the Hamiltonian (VII.14) and the constraints (VII.16) and (VII.21) are invariant, in the sense that their symmetry transformation vanishes on the hypersurface of phase space determined by the constraints. Let us state a series of results and properties of our gauge generator.

a) G(t) is a conserved quantity

$$
\begin{aligned}
\frac{d}{dt}G(t) &= \{G(t), H(t)\} \;+\; \frac{\partial}{\partial t}G(t) &\text{(VII.45)}\\
&= \int d\sigma d\sigma' \Bigg[\mathcal{P}^j(t,\sigma)\left(\frac{\delta(\delta\mathcal{Q}_j(t,\sigma))}{\delta\mathcal{Q}_i(t,\sigma')}\mathcal{Q}_j{}'(t,\sigma') - \partial_\sigma\delta(\sigma-\sigma')\delta\mathcal{Q}_j(t,\sigma')\right.\\
&\qquad \left. + \frac{\delta(\delta\mathcal{Q}_j(t,\sigma))}{\delta\Lambda(t,\sigma')}\dot{\Lambda}(t,\sigma')\right) \;-\; \delta(t,\sigma)\;\left(\frac{\delta\mathcal{K}(t,\sigma)}{\delta\mathcal{Q}_i(t,\sigma')}\mathcal{Q}_i{}'(t,\sigma')\right.\\
&\qquad \left.\left. - \frac{\delta(\mathcal{L}(t,\sigma))}{\delta\mathcal{Q}_i(t,\sigma')}\delta\mathcal{Q}_i(t,\sigma') + \frac{\delta\mathcal{K}(t,\sigma)}{\delta\Lambda(t,\sigma')}\dot{\Lambda}(t,\sigma')\right)\right] \;=\; 0. &\text{(VII.46)}
\end{aligned}
$$

The last term of (VII.45) is an explicit t derivative through $\Lambda(t,\sigma)$. In order to show (VII.46) we need to use the symmetry condition (VII.42) and the condition on $\Lambda(t,\sigma)$ in (VII.43).

b) All the constraints are invariant under the symmetry transformations.

Let us show first the invariance of (VII.21), which is nothing but the invariance of the equations of motion, as was to expected for $G(t)$ generating a symmetry,

$$
\begin{aligned}
&\{\tilde{\varphi}^i(t,\sigma), G(t)\} = \int d\sigma'[\mathcal{P}^j(t,\sigma')\delta\mathcal{Q}_j(t,\sigma') - \delta(\sigma')\mathcal{K}(t,\sigma')]\}\\
&= \int d\sigma' d\sigma'' \frac{\delta^2\mathcal{L}(t,\sigma'')}{\delta\mathcal{Q}_j(t,\sigma')\delta\mathcal{Q}_i(t,\sigma)}\delta\mathcal{Q}_j(t,\sigma') = \int d\sigma' \frac{\delta\tilde{\varphi}^j(t,\sigma')}{\delta\mathcal{Q}_i(t,\sigma)}\delta\mathcal{Q}_j(t,\sigma')\\
&= -\int d\sigma' \tilde{\varphi}^j(t,\sigma')\frac{\delta(\delta\mathcal{Q}_j(t,\sigma'))}{\delta\mathcal{Q}_i(t,\sigma)} \approx 0, \qquad \text{(VII.47)}
\end{aligned}
$$

where we have used an identity obtained from (VII.42),

$$
\int d\sigma d\sigma' \mathcal{E}^j(t,\sigma,\sigma')\delta\mathcal{Q}_j(t,\sigma') = \int d\sigma' \tilde{\varphi}^j(t,\sigma')\delta\mathcal{Q}_j(t,\sigma') \;=\; 0. \qquad \text{(VII.48)}
$$

We now prove the invariance of the remaining constraint (VII.16). Using

(VII.42) and (VII.48),

$$\{\varphi^i(t,\sigma), G(t)\} =$$
$$= -\int d\sigma' \varphi^j(t,\sigma') \frac{\delta(\delta\mathcal{Q}_j(t,\sigma'))}{\delta\mathcal{Q}_i(t,\sigma)} - \int d\sigma' [\int d\sigma'' \chi(\sigma',-\sigma'')\mathcal{E}^j(t;\sigma'',\sigma') \frac{\delta(\delta\mathcal{Q}_j(t,\sigma'))}{\delta\mathcal{Q}_i(t,\sigma)}$$
$$-\delta(\sigma')\frac{\delta(\mathcal{K}(t,\sigma'))}{\delta\mathcal{Q}_i(t,\sigma)} + \int d\sigma'' \chi(\sigma,-\sigma'') \frac{\delta\mathcal{E}^i(t;\sigma'',\sigma)}{\delta\mathcal{Q}_j(t,\sigma')}\delta\mathcal{Q}_j(t,\sigma')]$$
$$= -\int d\sigma' \varphi^j(t,\sigma') \frac{\delta(\delta\mathcal{Q}_j(t,\sigma'))}{\delta\mathcal{Q}_i(t,\sigma)} + \int d\sigma' \chi(\sigma,-\sigma')\tilde{\varphi}^j(t,\sigma') \frac{\delta(\delta\mathcal{Q}_j(t,\sigma'))}{\delta\mathcal{Q}_i(t,\sigma)} \approx 0.$$

Thus we have shown that the constraint surface defined by $\varphi \approx \tilde{\varphi} \approx 0$ is invariant under the transformations generated by $G(t)$.

c) Our Hamiltonian (VII.14) is the generator of time translations.

Consider a non-local Lagrangian in (VII.1) that does not depend on t explicitly, so that time translation is a symmetry of the Lagrangian. To show that the generator of such a symmetry is our Hamiltonian H in (VII.14) and that it is conserved, we should simply show that we recover its expression (VII.14) from the general form of the generator (VII.41). Indeed, the Lagrangian changes as $\delta L^{non} = \varepsilon \dot{L}^{non}$ under a time translation $\delta q_i(t) = \varepsilon \dot{q}_i(t)$. The corresponding generator in the present formalism is then, using (VII.41)

$$G_H(t) = \int d\sigma \, [\, \mathcal{P}^i(t,\sigma)(\varepsilon\mathcal{Q}_i{}'(t,\sigma)) - \delta(\sigma)(\varepsilon\mathcal{L}(t,\sigma)) \,], \quad \text{(VII.49)}$$

which is ε times the Hamiltonian (VII.14), as we wanted to show. Then, property b) applied to this case brings us back something that we imposed in the construction of the formalism: the constraints hypersurface is stable under time evolution.

As our Hamiltonian in the $1+1$ theory is the generator of time translations, it should considered as giving the energy of the system. If we had started with a local theory, but still we had insisted on using the $1+1$ formalism, the system of constraints would the allow us to reduce to one dimension less and to recover the standard Hamiltonian of such local theory. We will explicitly show in the appendix D how this works for a $U(1)$ *commutative case*. Nevertheless, for a truly non local theory, there is no such a simplification and the phase space typically remains infinite dimensional. Our discussion then shows that it is our Hamiltonian (VII.14) the one that we should use for computing the energy of the system.

Summary:

We have constructed the Hamiltonian symmetry generators of a general non-local theory working in a d+1 dimensional space. In this formulation original gauge symmetries in d dimensions are rigid symmetries in the d+1 dimensional space. The next section is an illustration of how our formalism can be applied to a NC gauge theory, like the ones we discussed in the previous chapter.

VII.6 $U(1)$ non-commutative gauge theory

Let us apply all the machinery of the new Hamiltonian formalism to one of the nonlocal theories that we studied in chapter V: a $U(1)$ NC gauge theory in an electric background. The action is

$$S = -\frac{1}{4} \int d^d x \ \widehat{F}_{\mu\nu}\widehat{F}^{\mu\nu}, \tag{VII.50}$$

where $\widehat{F}_{\mu\nu}$ is the field strength of the $U(1)$ NC gauge potential $\widehat{A}_\mu$ defined by[5]

$$\widehat{F}_{\mu\nu} \quad = \quad \partial_\mu \widehat{A}_\nu - \partial_\nu \widehat{A}_\mu - i[\widehat{A}_\mu, \widehat{A}_\nu]. \tag{VII.51}$$

In this section, all commutators are defined using the NC *-product that we defined in (V.29), so that $[f, g] \equiv f * g - g * f$.

Here we are mainly interested in the most general case of *space-time* non-commutativity with $\theta^{0i} \neq 0$.[6] This means that the action (VII.50) contains an infinite number of time derivatives of the field $\widehat{A}$, and it is therefore nonlocal in time. Let us now obtain the main equations that we will need later to translate to the $d+1$ formalism.

1. The EL equation of motion is

$$\widehat{D}_\mu \widehat{F}^{\mu\nu} \quad = \quad 0, \tag{VII.52}$$

   where the covariant derivative is defined by $\widehat{D} = \partial - i[\widehat{A}, \ \ ]$.

2. The gauge transformation is

$$\delta \widehat{A}_\mu \quad = \quad \widehat{D}_\mu \widehat{\lambda} \tag{VII.53}$$

[5] Once again, we recall that we put "hats"on NC fields, see section V.2.3.

[6] A Hamiltonian formalism for the magnetic theory ($\theta^{0i} = 0$) was analyzed in [149].

and it satisfies a non-Abelian gauge algebra,

$$(\delta_{\widehat{\lambda}}\delta_{\widehat{\lambda}'} - \delta_{\widehat{\lambda}'}\delta_{\widehat{\lambda}})\widehat{A}_\mu \;=\; -i\widehat{D}_\mu[\widehat{\lambda},\widehat{\lambda}']. \qquad \text{(VII.54)}$$

Since the field strength transforms covariantly as $\delta\widehat{F}_{\mu\nu} = -i[\widehat{F}_{\mu\nu},\widehat{\lambda}]$, the Lagrangian density of (VII.50) transforms as

$$\delta(-\frac{1}{4}\,\widehat{F}_{\mu\nu}\widehat{F}^{\mu\nu}) \;=\; \frac{i}{2}\,[\widehat{F}_{\mu\nu},\widehat{\lambda}]\;\widehat{F}^{\mu\nu}. \qquad \text{(VII.55)}$$

Using $\int dx(f*g) = \int dx(fg)$ and the associativity of the $*$-product, (VII.55) becomes a total divergence, as was to be expected for (VII.53) being a symmetry. We have just proven that the action (VII.50) is invariant under the $U(1)$ NC transformations.

VII.6.1 Going to the $d+1$ formalism

Not only the Lagrangian (VII.50) is non-local, but also the NC gauge transformation (VII.53) is since, for electric backgrounds ($\theta^{0i} \neq 0$), it contains time derivatives of infinite order in $\widehat{\lambda}$. Let us now proceed to construct the Hamiltonian and the generator for the $U(1)$ NC theory using the formalism introduced in the last section.

- We associate a d+1 gauge potential[7] $\widehat{\mathcal{A}}_\mu(t,\sigma,\mathbf{x})$ to the d dimensional one $\widehat{A}_\mu(t,\mathbf{x})$.
- We regard t as the evolution "time". For convenience of notation, we relabel $\sigma \to x^0$. This is the same σ appearing in $q_i(t,\sigma)$ in the last section.
- The other $(d-1)$ spatial coordinates $\mathbf{x}$ correspond to the indices i of $q_i(t,\sigma)$. The signature of d+1 space is $(-,-,+,+,...,+)$.

As explained above, the aim will be to convert the original motion in time into motion in x^0 of $\widehat{\mathcal{A}}_\mu(t,x^0,\mathbf{x})$, but not into motion in t. Despite its signature, we will often refer to x^0 as another spatial coordinate, and we will reserve the name of *time* for t.

The canonical system equivalent to the non-local action (VII.50) is defined by the Hamiltonian (VII.14) and the two constraints, (VII.16) and (VII.21). For our present theory, the Hamiltonian reads

$$H(t) \;=\; \int d^dx\,[\widehat{\Pi}^\nu(t,x)\partial_{x^0}\widehat{\mathcal{A}}_\nu(t,x) - \delta(x^0)\mathcal{L}(t,x)], \qquad \text{(VII.56)}$$

[7] From now on we will use calligraphic letters for fields in the $d+1$ formalism.

where $\widehat{\Pi}^\nu$ is a momentum for $\widehat{\mathcal{A}}_\nu$ and

$$\mathcal{L}(t,x) = -\frac{1}{4}\,\widehat{\mathcal{F}}_{\mu\nu}(t,x)\widehat{\mathcal{F}}^{\mu\nu}(t,x), \tag{VII.57}$$
$$\widehat{\mathcal{F}}_{\mu\nu}(t,x) = \partial_\mu\widehat{\mathcal{A}}_\nu(t,x) - \partial_\nu\widehat{\mathcal{A}}_\mu(t,x) - i[\widehat{\mathcal{A}}_\mu(t,x),\widehat{\mathcal{A}}_\nu(t,x)] \tag{VII.58}$$

Note that by using (VII.18), *the $*$-product involves now only $x^\mu = (x^0,\mathbf{x})$ but it does not involve t.* Thus it contains spatial derivatives of infinite order but no time derivative. The same applies for the Hamiltonian, it contains no derivative with respect to t, and so it is a good phase-space quantity, a function of the canonical pairs $\{\widehat{\mathcal{A}}_\mu(t,x),\widehat{\Pi}^\mu(t,x)\}$ with Poisson bracket

$$\{\widehat{\mathcal{A}}_\mu(t,x),\widehat{\Pi}^\nu(t,x')\} = \delta_\mu{}^\nu\ \delta^{(d)}(x-x'). \tag{VII.59}$$

The momentum constraint (VII.16) is

$$\begin{aligned}\varphi^\nu(t,x) &= \widehat{\Pi}^\nu(t,x) + \int dy\ \chi(x^0,-y^0)\ \widehat{\mathcal{F}}^{\mu\nu}(t,y)\ \widehat{\mathcal{D}}^y_\mu\delta(x-y)\\ &= \widehat{\Pi}^\nu(t,x) + \delta(x^0)\widehat{\mathcal{F}}^{0\nu}(t,x) - \frac{i}{2}\Big(\epsilon(x^0)[\widehat{\mathcal{F}}^{\mu\nu},\widehat{\mathcal{A}}_\mu] - [\epsilon(x^0)\widehat{\mathcal{F}}^{\mu\nu},\widehat{\mathcal{A}}_\mu]\Big) \approx 0.\end{aligned}$$

while the constraint (VII.20), which followed from the consistency of the one just obtained, turns out to be

$$\tilde{\varphi}^\nu(t,x) = \widehat{\mathcal{D}}_\mu\widehat{\mathcal{F}}^{\mu\nu}(t,x) \approx 0. \tag{VII.60}$$

Note that, as expected for a theory with gauge invariance, these constraints are reducible since $\widehat{\mathcal{D}}_\mu\tilde{\varphi}^\mu \equiv 0$. The Hamilton equation (VII.18), which now reads

$$\partial_t\widehat{\mathcal{A}}_\mu(t,x) = \{\widehat{\mathcal{A}}_\mu(t,x),H(t)\} = \partial_{x^0}\widehat{\mathcal{A}}_\mu(t,x)\,, \tag{VII.61}$$

together with the identification (VII.15), $\widehat{\mathcal{A}}_\mu(t,x^\nu) = \widehat{A}_\mu(t+x^0,\mathbf{x})$, can be used in (VII.60) to recover the original equations of motion. Finally, since the Lagrangian of (VII.50) translational invariant, the Hamiltonian (VII.56), as well as the constraints (VII.60) and (VII.60), are conserved.

Let us now show how to compute the generator of the NC $U(1)$ transformation. We just apply (VII.41) to our case,

$$G[\widehat{\Lambda}] = \int d^dx[\ \widehat{\Pi}^\mu\delta\widehat{\mathcal{A}}_\mu - \delta(x^0)\ \mathcal{K}^0\ ], \tag{VII.62}$$

where the last term must be evaluated from the surface term appearing in the variation of the Lagrangian, *i.e.* we must read it from the RHS of

(VII.55). Instead of rewriting (VII.55) as ∂_μ of some k^μ, it is easier to integrate by parts the second term in (VII.62) as follows

$$\int d^dx[-\delta(x^0)\ \mathcal{K}^0\ ] = \int d^dx[\frac{\epsilon(x^0)}{2}\ \partial_\mu\mathcal{K}^\mu\ ] = \int d^dx[\frac{\epsilon(x^0)}{2}\ \delta\mathcal{L}\ ].$$

We can then plug immediately the RHS of (VII.55) (after the usual replacements needed to go to the $d+1$ formalism). The final expression for the $U(1)$ generator is then

$$G[\widehat{\Lambda}] \ = \ \int d^dx \left[\widehat{\Pi}^\mu \widehat{\mathcal{D}}_\mu \widehat{\Lambda} \ + \ \frac{i}{4}\epsilon(x^0)\widehat{\mathcal{F}}_{\mu\nu}\ [\widehat{\mathcal{F}}^{\mu\nu}, \widehat{\Lambda}]\right], \qquad \text{(VII.63)}$$

where, as discussed in (VII.43), $\widehat{\Lambda}(t, x^\mu)$ is an arbitrary function satisfying

$$\dot{\widehat{\Lambda}}(t, x^\mu) \ = \ \partial_{x^0}\widehat{\Lambda}(t, x^\mu)\,. \qquad \text{(VII.64)}$$

The generator can be expressed as a linear combination of the constraints,

$$G[\widehat{\Lambda}] \ = \ \int d^dx\ \widehat{\Lambda}\left[-(\widehat{\mathcal{D}}_\mu\varphi^\mu) - \delta(x^0)\tilde{\varphi}^0 + \frac{i}{2}\left(\epsilon(x^0)[\tilde{\varphi}^\nu, \widehat{\mathcal{A}}_\nu] - [\epsilon(x^0)\tilde{\varphi}^\nu, \widehat{\mathcal{A}}_\nu]\right)\right]. \qquad \text{(VII.65)}$$

The fact that the generator (VII.65) is a sum of constraints shows explicitly the conservation of the generator on the constraint surface. It is not hard to check that $G[\widehat{\Lambda}]$ is conserved

$$\frac{d}{dt}G[\widehat{\Lambda}] \ = \ \{G[\widehat{\Lambda}], H\} \ + \ \frac{\partial}{\partial t}G[\widehat{\Lambda}] \ = \ 0\,, \qquad \text{(VII.66)}$$

in agreement with (VII.46).

We will conclude this section by computing the energy of a given field configuration. For this, we need to isolate the part of the Hamiltonian which does not vanish in the constraint surface. It turns out that the Hamiltonian can be rewritten as

$$H \ = \ G[\widehat{\mathcal{A}}_0] \ + \ \int d^dx\ \varphi^i\ \widehat{\mathcal{F}}_{0i} \ + \ E_L, \qquad \text{(VII.67)}$$

where the first term is the $U(1)$ generator (VII.65) replacing the parameter $\widehat{\Lambda}$ by $\widehat{\mathcal{A}}_0$; it therefore vanishes in the constraint surface. The last term E_L

is then the only relevant piece. Its explicit form is

$$\begin{aligned} E_L &= \int d^d x\, \delta(x^0)\, \{\frac{1}{2}\widehat{\mathcal{F}}^2_{0i} + \frac{1}{4}\widehat{\mathcal{F}}^2_{ij}\} \\ &+\frac{i}{2}\int dx\, \widehat{\mathcal{A}}_0\{\frac{1}{2}[\widehat{\mathcal{F}}^{ij}, \epsilon(x^0)\widehat{\mathcal{F}}_{ij}] - [\widehat{\mathcal{F}}^{0i}, \epsilon(x^0)\widehat{\mathcal{F}}_{0i}]\} \\ &+\frac{i}{2}\int dx\, \widehat{\mathcal{A}}_j\{[\widehat{\mathcal{F}}_{0i}, \epsilon(x^0)\widehat{\mathcal{F}}^{ij}] - [\epsilon(x^0)\widehat{\mathcal{F}}_{0i}, \widehat{\mathcal{F}}^{ij}]\}. \end{aligned} \tag{VII.68}$$

This expression is useful, for example, to evaluate the energy of classical configurations of the theory. The two terms in the first line have the same form as the "energy"of the commutative $U(1)$ theory. The novelty are the last two lines, which are non-local contributions. They need to be taken into account except for two cases in which they identically vanish

1. in magnetic backgrounds $\theta^{0i} = 0$,
2. in t independent solutions of $\mathcal{A}_\mu$.

VII.7 Seiberg-Witten map, gauge generators and Hamiltonians

One of the advantages of having a well defined Hamiltonian and phase space formalism is that it allows us to apply the whole machinery that was developed in classical mechanics (canonical transformation, Hamilton-Jacobi equation, action-angle variables...). In this section we give a new interpretation to the Seiberg and Witten map that we discussed in section V.2.4. We will show that, in the d+1 formalism, it can be seen as a simple phase space canonical transformation. This makes explicit the physical equivalence of describing the action in terms ordinary and NC fields. By finding the corresponding generating functional, we will able to map quantities between both theories. In particular, we will show how the gauge generator and the Hamiltonian obtained in the previous section for the NC case are mapped to those of the commutative theory.

In the following discussions we keep terms only up to the first order in θ. All equations implicitly omit any higher powers. The problem of finding exact results is just technical and it is probably not too difficult since it just boils down to finding the exact generating functional.

We recall from section V.2.4 that the SW map among quantities in both pictures looks like

$$\widehat{A}_\mu = A_\mu + \frac{1}{2}\theta^{\rho\sigma} A_\sigma (2\partial_\rho A_\mu - \partial_\mu A_\rho)\,, \qquad \text{(VII.69)}$$

$$\widehat{F}_{\mu\nu} = F_{\mu\nu} + \theta^{\rho\sigma} F_{\rho\mu} F_{\sigma\nu} - \theta^{\rho\sigma} A_\rho \partial_\sigma F_{\mu\nu}\,, \qquad \text{(VII.70)}$$

$$\widehat{\lambda}(\lambda, A) = \lambda + \frac{1}{2}\theta^{\rho\sigma} A_\sigma \partial_\rho \lambda\,, \qquad \text{(VII.71)}$$

so that under a commutative $U(1)$ transformation of $\delta A_\mu = \partial_\mu \lambda$, the mapped $\widehat{A}_\mu$ transforms as

$$\delta\widehat{A}_\mu = \partial_\mu\{\lambda + \frac{1}{2}\theta^{\rho\sigma} A_\sigma \partial_\rho \lambda\} + \theta^{\rho\sigma}\partial_\sigma\lambda\partial_\rho A_\mu \;=\; \widehat{D}_\mu \widehat{\lambda}. \qquad \text{(VII.72)}$$

Let us proceed to the d+1 formalism. As in the previous section, we denote the d+1 dimensional potentials $\widehat{\mathcal{A}}_\mu(t, x)$ and $\mathcal{A}_\mu(t, x)$ corresponding to d dimensional ones $\widehat{A}_\mu(t, \mathbf{x})$ and $A_\mu(t, \mathbf{x})$ respectively.[8] The way to realize the SW map as a canonical transformation is by means of the following generating function

$$W(\mathcal{A}, \widehat{\Pi}) = \int dx\, \widehat{\Pi}^\mu \left[\mathcal{A}_\mu + \frac{1}{2}\theta^{\rho\sigma}\mathcal{A}_\sigma(2\partial_\rho \mathcal{A}_\mu - \partial_\mu \mathcal{A}_\rho)\right] \;+\; W^0(\mathcal{A}), \qquad \text{(VII.73)}$$

where $W^0(\mathcal{A})$ is an arbitrary function of $\mathcal{A}_\mu$ of order θ. Any choice of $W^0(\mathcal{A})$ leads to the correct transformation of $\mathcal{A}_\mu$ as in (VII.69)

$$\widehat{\mathcal{A}}_\mu = \mathcal{A}_\mu + \frac{1}{2}\theta^{\rho\sigma}\mathcal{A}_\sigma(2\partial_\rho \mathcal{A}_\mu - \partial_\mu \mathcal{A}_\rho)\,. \qquad \text{(VII.74)}$$

At the same time, the canonical transformation determines the relation between Π^μ and $\widehat{\Pi}^\mu$, conjugate momenta of $\mathcal{A}_\mu$ and $\widehat{\mathcal{A}}_\mu$ respectively,

$$\begin{aligned}\Pi^\mu &= \widehat{\Pi}^\mu + \frac{1}{2}\widehat{\Pi}^\sigma\theta^{\rho\mu}(2\partial_\rho\mathcal{A}_\sigma - \partial_\sigma\mathcal{A}_\rho)\\ &- \partial_\rho(\theta^{\rho\sigma}\mathcal{A}_\sigma\widehat{\Pi}^\mu) + \frac{1}{2}\partial_\rho(\widehat{\Pi}^\rho\theta^{\mu\sigma}\mathcal{A}_\sigma) + \frac{\delta W^0(\mathcal{A})}{\delta\mathcal{A}_\mu},\end{aligned}$$

which can be inverted, to first order in θ,

$$\begin{aligned}\widehat{\Pi}^\mu &= \Pi^\mu + \theta^{\mu\rho}\Pi^\sigma\mathcal{F}_{\rho\sigma} + \Pi^\mu\frac{1}{2}\theta^{\rho\sigma}\mathcal{F}_{\rho\sigma}\\ &+ \theta^{\rho\sigma}\mathcal{A}_\sigma\partial_\rho\Pi^\mu - \frac{1}{2}(\partial_\rho\Pi^\rho)\theta^{\mu\sigma}\mathcal{A}_\sigma - \frac{\delta W^0(\mathcal{A})}{\delta\mathcal{A}_\mu}.\end{aligned} \qquad \text{(VII.75)}$$

[8] Remember, hats for fields in the non-commutative theory, and calligraphic letters for fields in the $d+1$ formalism

Note that the canonical transformation, (VII.74) and (VII.75), is independent of the concrete theories we are considering as we have not even needed to specify the action yet. Now that the canonical transformation is defined, we can use it to translate any phase space function from one picture to another.

Mapping the gauge generator.

We obtained the gauge generator for the NC $U(1)$ theory in (VII.63). Let us apply the canonical transformation to map it to the commutative picture. It is straightforward to show that, if we choose $W^0(\mathcal{A}) = 0$, the first part of the gauge generator is mapped to

$$\int dx[\ \widehat{\Pi}^\mu \widehat{\mathcal{D}}_\mu \widehat{\Lambda}(\Lambda, \mathcal{A})\ ] \;\; = \;\; \int dx\ [\ \Pi^\mu \partial_\mu \Lambda\ ], \tag{VII.76}$$

where

$$\widehat{\Lambda}(\Lambda, \mathcal{A}) = \Lambda + \frac{1}{2}\theta^{\rho\sigma}\mathcal{A}_\sigma \partial_\rho \Lambda, \qquad\qquad \dot{\Lambda} = \partial_{x^0}\Lambda. \tag{VII.77}$$

This result is again independent of the specific form of Lagrangian for $U(1)$ NC and commutative gauge theories. However, the second part of the NC gauge generator, whose origin was a surface term in the action, does depend on the specific theory we are considering. For the $U(1)$ NC theory, this term is nothing but the Lagrangian dependent term in (VII.63), which expanded to first order in θ reads

$$\frac{i}{4}\int dx\ \epsilon(x^0)\widehat{\mathcal{F}}_{\mu\nu}[\widehat{\mathcal{F}}^{\mu\nu}, \widehat{\Lambda}] \;\; = \;\; \frac{1}{4}\int dx\ \delta(x^0)\theta^{0i}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\partial_i\Lambda. \tag{VII.78}$$

In this case the generator of $U(1)$ NC transformations can be mapped to the commutative one

$$\begin{aligned} G[\widehat{\Lambda}(\Lambda, \mathcal{A})] \;\; &= \;\; \int dx\ \{\Pi^0 \partial_0 \Lambda\ + (\Pi^i + \frac{1}{4}\delta(x^0)\theta^{0i}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu})\partial_i \Lambda\} \\ &\quad - \int dx \frac{\delta W^0(\mathcal{A})}{\delta \mathcal{A}_\mu}\partial_\mu \Lambda \, = \, \int dx\ [\ \Pi^\mu \partial_\mu \Lambda\ ] \end{aligned} \tag{VII.79}$$

if we choose the arbitrary function in the canonical transformation to be

$$W^0(\mathcal{A}) \;\; = \;\; \frac{1}{4}\int dx\ \delta(x^0)\ \theta^{0\mu}\mathcal{A}_\mu \mathcal{F}_{\rho\sigma}\mathcal{F}^{\rho\sigma}. \tag{VII.80}$$

The right hand side of (VII.79) is the well-known generator of the $U(1)$ commutative theory (see appendix D).

Mapping the Hamiltonian.

Now we would like to see what is the form of the commutative $U(1)$ Hamiltonian obtained from the NC one (VII.56) under the SW map, (VII.74) and (VII.75). A short calculation yields

$$H_{com} = \int dx \; [\Pi^\nu(t,x)\mathcal{A}'_\nu(t,x) \; - \; \delta(x^0)\mathcal{L}_{com}(t,x)] \,, \tag{VII.81}$$

where

$$\mathcal{L}_{com}(t,x) = -\frac{1}{4}\mathcal{F}^{\nu\mu}\mathcal{F}_{\nu\mu} - \frac{1}{2}\mathcal{F}^{\mu\nu}\theta^{\rho\sigma}\mathcal{F}_{\rho\mu}\mathcal{F}_{\sigma\nu} + \frac{1}{8}\theta^{\nu\mu}\mathcal{F}_{\nu\mu}\mathcal{F}_{\rho\sigma}\mathcal{F}^{\rho\sigma}. \tag{VII.82}$$

Let us pause for a second. What should we have expected to obtain by this mapping? As we discussed in section V.2.4, commutative fields are ruled by the usual commutative product of functions and the usual gauge transformations. However, in their effective actions one has the annoying presence everywhere of the B-field, whose relation to θ in the decoupling limit is $B = \theta^{-1}$. Can we recover this point of view from the commutative U(1) Hamiltonian that we have obtain?

The answer is affirmative. The expression (VII.81) is nothing but the $d{+}1$ dimensional Hamiltonian that we would have obtained from an abelian $U(1)$ gauge theory with Lagrangian

$$L_{com}(t,\mathbf{x}) = -\frac{1}{4}F^{\nu\mu}F_{\nu\mu} - \frac{1}{2}F^{\mu\nu}\theta^{\rho\sigma}F_{\rho\mu}F_{\sigma\nu} + \frac{1}{8}\theta^{\nu\mu}F_{\nu\mu}F_{\rho\sigma}F^{\rho\sigma}, \tag{VII.83}$$

in d dimensions. And, as expected, one can check that this Lagrangian is, up to a total derivative, the expansion of the usual Born-Infield action to order F^3

$$L_{com} \;\; \sim \;\; -\sqrt{-\det(\eta_{\mu\nu} - \theta_{\mu\nu} + F_{\mu\nu})}\,|_{O(F^3)} \,. \tag{VII.84}$$

VII.8 BRST symmetry

In what follows, we will complete our exploitation of the $d{+}1$ formalism and setup future studies of nonlocal theories by means of some useful tools that were developed for local ones, concentrating on their BRST and field-antifield properties. The already familiar NC $U(1)$ theory will serve as an example all the way through. So, in what follows, one will find

- A study of the BRST symmetry [150][151] at classical and quantum levels. We will construct the BRST charge and the BRST invariant Hamiltonian working with the d+1 dimensional formulation, and we will check the nilpotency of the BRST generator.
- In order to map the BRST charges and Hamiltonians of the commutative and NC $U(1)$ gauge theories, we will generalize the SW map to a canonical transformation in the superphase space.
- In the last section we will study the BRST symmetry at Lagrangian level using the field-antifield formalism [152][153][9]. We will construct the solution of the classical master equation in the classical and gauge fixed basis. As this is a study at the Lagrangian level, the $d+1$ formalism will not be required. Still, we will be able to realize the SW map as an antibracket canonical transformation.

VII.8.1 Hamiltonian BRST charge

The BRST symmetry at classical level encodes the classical gauge structure through the nilpotency of the BRST transformations of the classical fields and ghosts [157][158][159]. BRST transformations of the classical fields are constructed from the original gauge transformation simply by changing the gauge parameters by ghost fields.

Let us consider again the $U(1)$ NC theory still in d dimensions. Its BRST transformations are then

$$\delta_B \widehat{A}_\mu \;=\; \widehat{D}_\mu \widehat{C}\,, \qquad \delta_B \widehat{C} \;=\; -i\widehat{C} * \widehat{C}, \tag{VII.85}$$

$$\delta_B \widehat{\overline{C}} \;=\; \widehat{B}\,, \qquad \delta_B \widehat{B} \;=\; 0, \tag{VII.86}$$

where $\widehat{C}, \widehat{\overline{C}}, \widehat{B}$ are the ghost, antighost and auxiliary field respectively. These are again a symmetry of the Lagrangian associated to (VII.50), since its change under the BRST transformations is

$$\delta_B L \;=\; \frac{i}{2}\,[\widehat{F}_{\mu\nu}, \widehat{C}]\;\widehat{F}^{\mu\nu}\,, \tag{VII.87}$$

which, as in (VII.55), can be shown to be a total divergence. To fix the gauge symmetry, we add a 'gauge fixing' term to the Lagrangian $\widehat{L}_{gf+FP}$ with the requirement that it must be of the form $\delta_B\widehat{\Psi}$. If we choose the gauge fixing fermion to be

$$\widehat{\Psi} = \widehat{\overline{C}}\;(\partial^\mu \widehat{A}_\mu + \alpha\widehat{B})\,, \tag{VII.88}$$

[9] See [154][155][156] for reviews on this subject.

then the $\widehat{L}_{gf+FP}$ is, up to a total derivative,

$$\widehat{L}_{gf+FP} \;=\; -\,\partial^\mu \widehat{\overline{C}}\;\widehat{D}_\mu \widehat{C} \;+\; \widehat{B}\;(\partial^\mu \widehat{A}_\mu + \alpha \widehat{B}). \tag{VII.89}$$

By construction, this term does not spoil the symmetry. Indeed

$$\delta_B \widehat{L}_{gf+FP} \;=\; \partial^\mu (\widehat{B}\widehat{D}_\mu \widehat{C}). \tag{VII.90}$$

Going to $d+1$.

In order to construct the generator of the BRST transformations and the BRST invariant Hamiltonian we should use the d+1 dimensional formulation. We use calligraphic characters for the d+1 dimensional fields $\widehat{\mathcal{C}}, \widehat{\overline{\mathcal{C}}}, \widehat{\mathcal{B}}$, corresponding to the d dimensional ones $\widehat{C}, \widehat{\overline{C}}, \widehat{B}$, respectively. The resulting BRST invariant Hamiltonian is given by

$$\begin{aligned}
H(t) \;&=\; H^{(0)} \;+\; H^{(1)}\,, \\
H^{(0)} \;&=\; \int dx\;[\widehat{\Pi}^\nu(t,x)\widehat{\mathcal{A}}'_\nu(t,x) \;+\; \widehat{\mathcal{P}_c}(t,x)\widehat{\mathcal{C}}'(t,x) \;-\; \delta(x^0)\widehat{\mathcal{L}}^0(t,x)], \\
H^{(1)} \;&=\; \int dx\;[\widehat{\mathcal{P}}_{\widehat{\mathcal{B}}}\widehat{\mathcal{B}}'(t,x) \;+\; \widehat{\mathcal{P}_{\overline{\mathcal{C}}}}(t,x)\widehat{\overline{\mathcal{C}}}'(t,x) \;-\; \delta(x^0)\widehat{\mathcal{L}}_{gf+FP}(t,x)]\,,
\end{aligned}$$

while the BRST charge is

$$Q_B \;=\; Q_B^{(0)} \;+\; Q_B^{(1)}\,, \tag{VII.91}$$

$$Q_B^{(0)} \;=\; \int dx \left[\widehat{\Pi}^\mu \widehat{\mathcal{D}}_\mu \widehat{\mathcal{C}} - i\widehat{\mathcal{P}_{\mathcal{C}}} * \widehat{\mathcal{C}} * \widehat{\mathcal{C}} + \frac{1}{2}\epsilon(x^0)\delta_B \widehat{\mathcal{L}}^0(t,x)\right], \tag{VII.92}$$

$$Q_B^{(1)} \;=\; \int dx \left[\,\widehat{\mathcal{P}_{\overline{\mathcal{C}}}}\;\widehat{\mathcal{B}} \;+\; \frac{1}{2}\epsilon(x^0)\delta_B \widehat{\mathcal{L}}_{gf+FP}(t,x)\right]. \tag{VII.93}$$

Q_B is an analogue of the BFV charge [160][161] for our NC $U(1)$ theory. $H^{(0)}, Q_B^{(0)}$ are the "gauge unfixed" and H, Q_B are "gauge fixed" Hamiltonians and BRST charges respectively. Finally, using the graded symplectic structure of the superphase space [162]

$$\begin{aligned}
\{\widehat{\mathcal{A}}_\mu(t,x), \widehat{\Pi}^\nu(t,x')\} \;&=\; \delta_\mu{}^\nu\;\delta^{(d)}(x-x'), \quad \{\widehat{\mathcal{C}}(t,x), \widehat{\mathcal{P}}_{\widehat{\mathcal{C}}}(t,x')\} \;=\; \delta^{(d)}(x-x'), \\
\{\widehat{\overline{\mathcal{C}}}(t,x), \widehat{\mathcal{P}}_{\widehat{\overline{\mathcal{C}}}}(t,x')\} \;&=\; \delta^{(d)}(x-x'), \qquad\;\; \{\widehat{\mathcal{B}}(t,x), \widehat{\mathcal{P}}_{\widehat{\mathcal{B}}}(t,x')\} \;=\; \delta^{(d)}(x-x'),
\end{aligned}$$

we have

$$\{H^{(0)}, Q_B^{(0)}\} \;=\; \{Q_B^{(0)}, Q_B^{(0)}\} \;=\; 0,$$

and

$$\{H, Q_B\} \;=\; \{Q_B, Q_B\} \;=\; 0.$$

Thus the BRST charges are nilpotent and the Hamiltonians are BRST invariant both at the gauge unfixed and the gauge fixed levels.

VII.8.2 Seiberg-Witten map in superphase space

Now we would like to see how the BRST charges and the BRST invariant Hamiltonians of the NC and commutative gauge theories are related. In order to do that we will extend the SW map to a canonical transformation in the superphase space $\{\mathcal{A}, \mathcal{C}, \overline{\mathcal{C}}, \mathcal{B}, \mathit{\Pi}, \mathcal{P}_{\mathcal{C}}, \mathcal{P}_{\overline{\mathcal{C}}}, \mathcal{P}_{\mathcal{B}}\}$. We introduce the generating function

$$\begin{aligned} W(\mathcal{A}, \mathcal{C}, \overline{\mathcal{C}}, \mathcal{B}, \widehat{\mathit{\Pi}}, \widehat{\mathcal{P}}_{\mathcal{C}}, \widehat{\mathcal{P}}_{\overline{\mathcal{C}}}, \widehat{\mathcal{P}}_{\mathcal{B}}) &= \int dx \, \left[\widehat{\mathit{\Pi}}^{\mu} \left(\mathcal{A}_\mu + \frac{1}{2}\theta^{\rho\sigma}\mathcal{A}_\sigma(2\partial_\rho\mathcal{A}_\mu - \partial_\mu\mathcal{A}_\rho) \right) \right. \\ &+ \left. \widehat{\mathcal{P}}_{\mathcal{C}} \left(\mathcal{C} + \frac{1}{2}\theta^{\rho\sigma}\mathcal{A}_\sigma\partial_\rho\mathcal{C} \right) \; + \; \widehat{\mathcal{P}}_{\overline{\mathcal{C}}}\overline{\mathcal{C}} \; + \; \widehat{\mathcal{P}}_{\mathcal{B}}\mathcal{B} \right] \\ &+ \; W^0(\mathcal{A}, \mathcal{C}) \; + \; W^1(\mathcal{A}, \mathcal{C}, \overline{\mathcal{C}}, \mathcal{B}). \end{aligned}$$

As before, $W^0(\mathcal{A}, \mathcal{C})$ depends on the specific form of the Lagrangian and the novelty is the appearance of $W^1(\mathcal{A}, \mathcal{C}, \overline{\mathcal{C}}, B)$, whose form depends also on the gauge fixing. For the $U(1)$ NC theory and for the gauge fixing (VII.88), we must choose

$$W^0(\mathcal{A}, \mathcal{C}) \; = \; \frac{1}{4}\int dx \; \delta(x^0) \; \theta^{0\mu}\mathcal{A}_\mu\mathcal{F}_{\rho\sigma}\mathcal{F}^{\rho\sigma} \tag{VII.94}$$

as in (VII.80) and

$$\begin{aligned} W^1(\mathcal{A}, \mathcal{C}, \overline{\mathcal{C}}, \mathcal{B}) \; &= \; \int dx \; \frac{1}{2}\epsilon(x^0) \left[\partial^\mu \{ \frac{1}{2}\theta^{\rho\sigma}\mathcal{A}_\sigma(2\partial_\rho\mathcal{A}_\mu - \partial_\mu\mathcal{A}_\rho) \}\mathcal{B} \right. \\ &+ \; \left. \{ \frac{1}{2}\theta^{\rho\sigma}\mathcal{A}_\sigma(2\partial_\rho\mathcal{A}_\mu - \partial_\mu\mathcal{A}_\rho)\partial_\sigma\mathcal{C} + \frac{1}{2}\theta^{\rho\sigma}\mathcal{A}_\sigma\partial_\mu\partial_\rho\mathcal{C} \; \} \; \partial^\mu\overline{\mathcal{C}} \; \right]. \end{aligned}$$

The transformations are obtained from the generating function by

$$\widehat{\Phi}^A = \frac{\partial_l W}{\partial \widehat{P}_A}, \qquad P_A = \frac{\partial_r W}{\partial \Phi^A}, \tag{VII.95}$$

where Φ^A represent any fields, P_A their conjugate momenta, and ∂_r and ∂_l are right and left derivatives respectively. Explicitly we have

$$\widehat{\mathcal{A}}_\mu \; = \; \mathcal{A}_\mu + \frac{1}{2}\theta^{\rho\sigma}\mathcal{A}_\sigma(2\partial_\rho\mathcal{A}_\mu - \partial_\mu\mathcal{A}_\rho), \tag{VII.96}$$

$$\widehat{\mathcal{C}} \; = \; \mathcal{C} + \frac{1}{2}\theta^{\rho\sigma}\mathcal{A}_\sigma\partial_\rho\mathcal{C}, \tag{VII.97}$$

$$\widehat{\overline{\mathcal{C}}} \; = \; \overline{\mathcal{C}}, \tag{VII.98}$$

$$\widehat{\mathcal{B}} \; = \; \mathcal{B}, \tag{VII.99}$$

and

$$\begin{aligned}
\widehat{\Pi}^\mu &= \Pi^\mu + \theta^{\mu\rho}\Pi^\sigma\mathcal{F}_{\rho\sigma} + \Pi^\mu\frac{1}{2}\theta^{\rho\sigma}\mathcal{F}_{\rho\sigma} + \theta^{\rho\sigma}\mathcal{A}_\sigma\partial_\rho\Pi^\mu - \frac{1}{2}(\partial_\rho\Pi^\rho)\theta^{\mu\sigma}\mathcal{A}_\sigma \\
&\quad + \frac{1}{2}\mathcal{P}_{\mathcal{C}}\theta^{\mu\sigma}\partial_\sigma\mathcal{C} \ - \ \frac{\delta(W^0+W^1)}{\delta\mathcal{A}_\mu}, \qquad \text{(VII.100)}\\
\widehat{\mathcal{P}}_{\mathcal{C}} &= \mathcal{P}_{\mathcal{C}} + \frac{1}{2}\theta^{\rho\sigma}\partial_\rho(\mathcal{P}_{\mathcal{C}}\mathcal{A}_\sigma) \ - \ \frac{\delta_r(W^0+W^1)}{\delta\mathcal{C}}, \qquad \text{(VII.101)}\\
\widehat{\mathcal{P}}_{\overline{\mathcal{C}}} &= \mathcal{P}_{\overline{\mathcal{C}}} \ - \ \frac{\delta_r W^1}{\delta\overline{\mathcal{C}}}, \qquad \text{(VII.102)}\\
\widehat{\mathcal{P}}_{\mathcal{B}} &= \mathcal{P}_{\mathcal{B}} \ - \ \frac{\delta_r W^1}{\delta\mathcal{B}}. \qquad \text{(VII.103)}
\end{aligned}$$

These transformations allow us to map the NC quantities to the commutative ones.

Mapping the BRST charge.
The NC BRST charge (VII.91) becomes

$$Q_B \ = \ Q_B^{(0)} \ + \ Q_B^{(1)} \ = \ \int dx[\Pi^\mu\partial_\mu\mathcal{C} \ + \ \mathcal{P}_{\overline{\mathcal{C}}}\mathcal{B} \ - \ \delta(x^0)\mathcal{B}\partial^0\mathcal{C}\ ] \qquad \text{(VII.104)}$$

$$= \ \int dx[\Pi^\mu\partial_\mu\mathcal{C} \ + \ \mathcal{P}_{\overline{\mathcal{C}}}\mathcal{B} \ + \ \frac{1}{2}\epsilon(x^0)\delta_B\mathcal{L}_{gf+FP}(t,x)\ ], \quad \text{(VII.105)}$$

where $\mathcal{L}_{gf+FP}(t,x)$ is the abelian gauge fixing Lagrangian

$$\mathcal{L}_{gf+FP} \ = \ -\partial^\mu\overline{\mathcal{C}}\ \partial_\mu\mathcal{C} \ + \ \mathcal{B}\ (\partial^\mu\mathcal{A}_\mu + \alpha\mathcal{B}). \qquad \text{(VII.106)}$$

Mapping the Hamiltonian.
The total NC $U(1)$ Hamiltonian (VII.91) becomes

$$H = \int dx\ [\Pi^\nu\mathcal{A}'_\nu + \mathcal{P}_{\mathcal{C}}\mathcal{C}' + \mathcal{P}_{\overline{\mathcal{C}}}\overline{\mathcal{C}}' + P_{\mathcal{B}}\mathcal{B}' - \delta(x^0)(\mathcal{L}_{com} + \mathcal{L}_{gf+FP})]. \quad \text{(VII.107)}$$

Remember that $\mathcal{L}_{com}$ is the $U(1)$ Lagrangian in the commutative given in (VII.82).

Summarizing, we have been successful in mapping the NC and commutative charges in the $d+1$ formalism by generalizing the SW map to a canonical transformation in the superphase space.

VII.8.3 Field-antifield formalism for $U(1)$ non-commutative theory

The field-antifield formalism allows us to study the BRST symmetry of a general gauge theory by introducing a canonical structure at a Lagrangian level [152][153][154][155]. The classical master equation in the classical basis encodes the gauge structure of the generic gauge theory [158][159]. The solution of the classical master equation in the gauge fixed basis gives the "quantum action" to be used in the path integral quantization. Any two solutions of the classical master equations are related by a canonical transformation in the antibracket sense [153].

Here we will apply these ideas to the $U(1)$ NC theory. Since we work at a Lagrangian level there is no need to go to the $d+1$ formalism, and we stay in d dimensions. In the classical basis the set of fields and antifields are

$$\Phi^A = \{\widehat{A}_\mu, \widehat{C}\}, \qquad \Phi^*_A = \{\widehat{A}^*_\mu, \widehat{C}^*\}. \tag{VII.108}$$

The solution of the classical master equation

$$(S, S) = 0, \tag{VII.109}$$

is given by[10]

$$S[\Phi, \Phi^*] = I[\widehat{A}] + \widehat{A}^*_\mu \widehat{D}^\mu \widehat{C} - i\widehat{C}^*(\widehat{C} * \widehat{C}), \tag{VII.110}$$

where $I[\widehat{A}]$ is the classical action and the antibracket $(\ ,\ )$ is defined by

$$(X, Y) = \frac{\partial_r X}{\partial \Phi^A}\frac{\partial_l Y}{\partial \Phi^*_A} - \frac{\partial_r X}{\partial \Phi^*_A}\frac{\partial_l Y}{\partial \Phi^A}. \tag{VII.111}$$

The gauge fixed basis can be analyzed by introducing the antighost and auxiliary fields and the corresponding antifields. It can be obtained from the classical basis by considering a canonical transformation in the antibracket sense

$$\begin{aligned} \Phi^A &\longrightarrow \Phi^A \\ \Phi^*_A &\longrightarrow \Phi^*_A + \frac{\partial_r \Psi}{\partial \Phi^A}\,, \end{aligned} \tag{VII.112}$$

which is generated by

$$\widehat{\Psi} = \widehat{\overline{C}}\ (\partial^\mu \widehat{A}_\mu + \alpha \widehat{B}), \tag{VII.113}$$

[10] As usual in the antifield formalism, d dimensional integration is understood in summations.

where $\widehat{\overline{C}}$ is the antighost and $\widehat{B}$ is the auxiliary field. In the new gauge fixed basis, we then have

$$S[\Phi, \Phi^*] = \widehat{I}_\Psi + \widehat{A}^{*\mu} \widehat{D}_\mu \widehat{C} - i\widehat{C}^*(\widehat{C} * \widehat{C}) + \widehat{\overline{C}}^* \widehat{B}, \tag{VII.114}$$

where $\widehat{I}_\Psi$ is the "quantum action", given by

$$\widehat{I}_\Psi = I[\widehat{A}] + (-\partial_\mu \widehat{\overline{C}}\ \widehat{D}^\mu \widehat{C} + \widehat{B}\ \partial_\mu \widehat{A}^\mu + \alpha \widehat{B}^2). \tag{VII.115}$$

The action $\widehat{I}_\Psi$ has well defined propagators and is the starting point of the Feynman perturbative calculations.

Now we would like to study what is the SW map in the space of fields and antifields. We first consider it in the classical basis. In order to do that we construct a canonical transformation in the antibracket sense

$$\widehat{\Phi}^A = \frac{\partial_l F_{cl}[\Phi, \widehat{\Phi}^*]}{\partial \widehat{\Phi}^*_A}, \quad \Phi^*_A = \frac{\partial_r F_{cl}[\Phi, \widehat{\Phi}^*]}{\partial \Phi^A}, \tag{VII.116}$$

where

$$F_{cl} = \widehat{A}^{*\mu}\left(A_\mu + \frac{1}{2}\theta^{\rho\sigma} A_\sigma (2\partial_\rho A_\mu - \partial_\mu A_\rho)\right) + \widehat{C}^*(C + \frac{1}{2}\theta^{\rho\sigma} A_\sigma \partial_\rho C). \tag{VII.117}$$

The NC and commutative gauge structures are then are mapped to each other

$$\widehat{A}^*_\mu \widehat{D}^\mu \widehat{C} - i\widehat{C}^*(\widehat{C} * \widehat{C}) = A^*_\mu \partial^\mu C. \tag{VII.118}$$

We can generalize the previous results to the gauge fixed basis. In this case the transformations of the antighost and the auxiliary field sectors should be taken into account. The generator of the canonical transformation is modified from (VII.117) to

$$F_{gf} = F_{cl} + \left(\widehat{\overline{C}}^* + \frac{1}{2}\theta^{\rho\sigma}\partial^\mu \left(A_\sigma (2\partial_\rho A_\mu - \partial_\mu A_\rho)\right)\right) \overline{C} + \widehat{B}^* B. \tag{VII.119}$$

Note that the additional term gives rise to new terms in $A^{*\mu}$ and $\overline{C}^*$ while the others remain the same as in the classical basis. In particular

$$\widehat{\overline{C}} \ = \ \overline{C}, \qquad \widehat{B} \ = \ B. \tag{VII.120}$$

Using the canonical transformation we can express (VII.114) and (VII.115) as

$$S[\Phi, \Phi^*] = I_\Psi + A^{*\mu}\partial_\mu C + \overline{C}^* B\,, \tag{VII.121}$$

where

$$I_\Psi = I[\widehat{A}(A)] + (-\partial_\mu \overline{C}\ \partial^\mu C + B\ \partial_\mu A^\mu + \alpha B^2)\,, \tag{VII.122}$$

and $I[\widehat{A}(A)]$ is the classical action in terms of A_μ. This is indeed the quantum action for the commutative $U(1)$ BRST invariant action in the gauge fixed basis. In this way the canonical transformation (VII.119) maps the $U(1)$ NC structure of the $S[\Phi, \Phi^*]$ into the commutative one in the gauge fixed basis.

VII.9 Discussion

In this chapter the Hamiltonian formalism of the non-local theories has been discussed by using a d+1 dimensional formulation [29][30]. For any given non-local Lagrangian in d dimensions the corresponding Hamiltonian in d+1 is defined in (VII.14). The equivalence with the original non-local theory is ensured by imposing two constraints (VII.16) and (VII.21) consistent with the time evolution. The degrees of freedom of the extra dimension (denoted by the coordinate σ) have their origin in the infinite degrees of freedom associated with the non-locality. The fact that we have been led to a theory with "two times" should be intimately related to their acausality [28][163] and non-unitarity [27][164]. It remains however as an interesting exception the case of light-like NC theories.

The $d + 1$ formalism is also applicable to *local* and higher derivative theories. In these cases the set of constraints can be used to reduce the redundant degrees of freedom of the infinite dimensional phase space, reproducing the standard d dimensional formulations [30].

We have analyzed the symmetry generators of non-local theories in the Hamiltonian formalism. In particular, we have shown that the Hamiltonian is the conserved charge under time translations, justifying its interpretation as the energy of a configuration. We exemplified the formalism by applying it to the electric NC $U(1)$ gauge theory. We remark that gauge transformations in d dimensions are described as a rigid symmetry in d+1 dimensions. The generators of *rigid* transformations in d+1 dimensions turn out to be the generators of *gauge* transformations when the reduction to d dimensions can be performed as is shown for the $U(1)$ commutative gauge theory in the appendix D.

Within this formalism, we reinterpreted the Seiberg-Witten map as a canonical transformation. This allowed us to map the Hamiltonians and the gauge generators of NC and commutative theories. We exemplified this

by explicitly mapping the $U(1)$ NC action to the commutative DBI one to order F^3.

The BRST symmetry has been analyzed at Hamiltonian and Lagrangian levels, and functionals in the commutative and NC pictures have been mapped by extending the SW map to a canonical transformation of the ghosts in the super phase space. Purely at a Lagrangian level, using the field-antifield formalism, we have seen how the solution of the classical master equation for NC and commutative theories are related by a canonical transformation in the antibracket sense. This result shows that the antibracket cohomology classes of both theories coincide in the space of non-local functionals. The explicit forms of the antibracket canonical transformations could be useful to study the observables, anomalies, etc. in the $U(1)$ NC theory.

VIII. CONCLUSIONS

In this thesis we have performed a little tour about two of the main open branches of String Theory that followed the discovery of D-branes: the gauge/string correspondence and noncommutative theories. At the end we managed to somehow close the loop and link them two, trying to shed new light on controversial issues like the UV/IR mixing of NC theories from the point of view of its closed string dual.

During the tour we stopped at some points that we found interesting to be studied on their own. One of the issues that we needed to face repeatedly was how to stabilize a D-brane when the target space or the embedding are not flat. During the thesis we have seen at least four qualitatively different ways to achieve it:

- Either in flat space or in a target space of the style $\mathbb{R}^{1,1}\times\mathcal{M}_8$, with $\mathcal{M}_8$ any special holonomy manifold, a D2 brane can be stabilized in a tubular shape with an arbitrary cross section. In the case that $\mathcal{M}_8$ is not completely flat, the only requirement is that the longitudinal direction lives in the $\mathbb{R}$ factor. The result confirms the understanding that the Poynting vector generated by the electromagnetic fields (or, equivalently, the local density of F1 and D0-branes) can be chosen to locally compensate not only the D2 tension, but also the extra gravitational effect produced by the curvature of $\mathcal{M}_8$. Furthermore, the D-brane picture allowed us to find new supergravity solutions of type IIA which describe the backreaction of these generalized supertubes, providing backgrounds whose preserved fraction of supersymmetry ranges from 1/32 to 1/8.

- In a completely counterintuitive manner, the D-brane worldvolume can be an $AdS_{p'}\times\Sigma^{q'}$ manifold of an $AdS_p\times S^q$ background, with $\Sigma^{q'}$ a minimal submanifold of the S^q. In all the cases we studied, $\Sigma^{q'}$ had actually the maximal volume within its homology class, and there was no topological obstruction at all for $\Sigma^{q'}$ to collapse into a point. The stability of all the examples considered could be understood by reduc-

ing the worldvolume theory on $\Sigma^{q'}$ and checking in the effective Lagrangian in $AdS_{p'}$ if there was any tachyonic fluctuation violating the corresponding Breitenlohner-Freedman bound. In most cases, stability follows from supersymmetry, as most of the stable embeddings can be understood as arising from supersymmetric brane configurations in flat space in which on type of branes is replaced by its background. We exploited this mechanism to try to embed branes in a stable but non-supersymmetric way, with the aim to provide holographic duals of superconformal field theories in which non-supersymmetric matter is added. We mentioned however that this is work in progress and that we can not be conclusive about the success of this possibility yet.

- Angular momentum alone can stabilize a relativistic surface of any dimension, as long as the dimension of target space is large enough. Although in section III.4.1 we have just exploited this for strings in type IIB, by S-duality we would obtain a D1, and by T-dualities we would obtain a general Dp-brane with one compact rotating direction. Applying the results of [73], more directions could become involved in the rotation as long as they form a minimal surface within the corresponding sphere. It is remarkable that this phenomenon already happens in flat space where it is perturbatively stable [74] and completely independent of supersymmetry.

- The last possibility is maybe the more intuitive: D-branes can wrap calibrated cycles of special holonomy manifolds and still preserve some supersymmetry. The cycles prevent the collapse of the brane in a fine-tuned manner, as they realize geometrically the twisting of the gauge theory that lives on the worldvolume. First we reviewed that the holonomy must be 'special' in order for the target space to preserve susy and the cycle must be 'calibrated' in order for some supersymmetry to be linearly realized in its worldvolume.

Let us expand on the latter possibility. The understanding of the brane/target-space setup provided the necessary intuition, together with the technical improvement provided by gauged supergravities, to construct the string dual of an $SU(N)$ $\mathcal{N}=2$ SYM theory in 2+1. One of the nice points about this construction is that it involved D6-branes. In this particular duality, they wrap a four-dimensional Kähler cycle Σ_4 in a Calabi-Yau threefold. The following two facts

- D6-brane solutions without extra fluxes lift to a purely gravitational 11d backgrounds,

- our particular D6 configuration has a 2+1 flat noncompact part, so that in 11d the solution is $\mathbb{R}^{1,2} \times X_8$,

imply that X_8 has to be a special holonomy manifold. As the number of preserved supersymmetries is 4, the table in page 112 implies that X_8 has to be a CY_4. In other words, the cycle, its transverse space and the M-theory circle have to conspire in order to produce a noncompact 8d manifold with $SU(4)$ holonomy. Our particular CY_4 (IV.92)-(IV.93) is of a conical type with constant radial sections being a $U(1)$ fibration over an $S^2 \times \Sigma_4$ base. The complete analytic solution of the BPS equations involve an integration constant that essentially measures the radius of the S^2 at the origin. Unless this constant is set to zero, the metric does not suffer from a conical singularity at the origin and it turns out to be locally regular everywhere.

Having obtained the desired 11d metric we compactified along a circle to analyze the problem in type IIA. There we performed a probe analysis in order to obtain the moduli space in the Coulomb branch of the dual field theory and we found that it was a 2d Kähler space as naively expected for an $\mathcal{N} = 2$ gauge theory in 2+1. We recall that although it has long been known that instanton contributions completely destroy the moduli space, their effects are exponentially suppressed with N. They are then expected to be invisible in the supergravity approximation in agreement with our result. Indeed, the shape of the moduli space that we get from supergravity resembles very much the classical plus 1-loop moduli space (see fig. IV.4 in page 142) that is obtained from the field theory. The latter is known to have a singularity as the vev of the scalar approaches the bare coupling, a sign that more loops should be taken into account. The probe result seems to be able to smooth this singularity and push it to the origin, where the vev of the scalar is zero. At this point supergravity is not valid as the curvature of the background diverges. However, we can approach as much as required by taking $g_s N$ large enough. It could be that supergravity is giving us the all-loops correction to the moduli space, although this is just a suggestive possibility.

Purely within the AdS/CFT correspondence we considered in detail the possibility of testing the duality in sectors where supersymmetry could be absent. The GKP ideas have allowed to produce a number shortcuts to bypass the quantization of the σ-model in various RR-backgrounds. Essentially one just needs to identify the appropriate σ-model soliton that carries a set of charges and compute their classical value. When these numbers are large, the quantum σ-model corrections can happen to be small, yielding a

non-perturbative prediction for the dual field theory operators.

A set of qualitatively new solutions are strings that rotate in the S^5 with three angular momenta. The results that the GKP method provides here are surprising as they show an exact agreement with perturbative computations in the CFT side. When all momenta are large, the string states (or its dual operators) are very far from the BMN vacuum (or the BMN operators), so they were originally thought be providing tests of the AdS/CFT 'far from supersymmetry'.

We have shown however that precisely at large momenta the strings approach another set of supersymmetric states, preserving 1/8 or 1/4 supersymmetry for 3 or 2 non-vanishing momenta respectively. We have proven this result both from a worldsheet κ-symmetry computation and from a standard BPS argument that follows from the background $PSU(2,2|4)$ superalgebra. Asymptotic restoration of supersymmetry suggests, among other things, that all these configurations must become stable in the limit, and this can be proven perturbatively by noting that the mass of all the tachyonic modes (in the unstable cases) go to zero in the limit.

We noted that, exactly as in the BMN case, the exact supersymmetric string is one for which all its energy is due to rotation; this means that the string becomes effectively tensionless. That the large angular momentum limit is also a tensionless limit is a non-trivial feature at all. In flat space, an increase of the angular momentum is always accompanied by an increase of the length of the soliton in such a way that the kinetic energy remains comparable to the energy due to the tension. One might suspect that all we need to do is to put the string to rotate in a compact space, just like the S^5, as the length of the string soon reaches a maximum value. But even this intuition can fail. There is no reason why the soliton should be able to absorb the extra energy by speeding up without growing. Giant gravitons provide an example: the giant D3-branes have an upper bound on the angular momentum that corresponds to the point in which they have maximal volume in the S^5; beyond that angular momentum the solution simple does not exist.

The spinning strings are very special this sense. Let us consider a string with only two equal angular momenta $J_1 = J_2 = J$ for simplicity, and consider an almost collapsed string with very low J. It is easy to see that, just as in flat space, J depends on an inverse power of the angular velocity ω, so that we need to decrease ω in order to increase J. As we do so the string starts expanding. This phase is qualitatively identical to the string rotating in flat space. However, there is a value of J for which the

string reaches its maximal size within the S^5. Beyond this critical point, the system suffers a kind of 'phase transition' in which J suddenly starts growing with ω. This is the second phase which is absent for giant gravitons and the responsible for allowing the string to absorb more and more kinetic energy until it effectively behaves as a tensionless string.

Somewhat as an aside, let us mention that there seems to be a paradox here. On the one hand, the action for a tensionless string is invariant under conformal rescalings of the background metric. On the other hand, we know that $AdS_5 \times S^5$ is conformal to flat 10d space. How do we explain that the string becomes supersymmetric in the former and not in the latter? The solution is to recall that $AdS_5 \times S^5$ is conformal but not *superconformal* to flat 10d space, as explained in [165].

The reason why the tensionless property is so relevant is that the perturbative field theory calculations always come in a power series of $\lambda = g_{YM}^2 N$. At best, they may acquire important combinatoric factors which soften the effective relevant coupling; this is the case of BMN and of the operators dual to the 3-angular momenta strings, where $\lambda_{eff} \sim \lambda/J^2$. Therefore, the field theory expansion is in terms of the ratio between the string tension and its kinetic energy. Any chance to test the duality by means of perturbative CFT computations must then involve an almost tensionless soliton in the string theory side.

As an open question, it remains to understand better if it is the supersymmetry or the tensionless property what really makes the comparison possible. As we discussed in detail in section III.4.7, one of the main points to explain is the fact that the σ-model quantum corrections do not spoil the classical result at any order. The pulsating string of [14] seems to provide an example of a successful matching for configurations arbitrarily far from being BPS which, nevertheless, become asymptotically tensionless. But as we discussed, they did not check the σ-model corrections and, at this stage, their matching could be a coincidence. The more recent results of [18], in which similar rotating strings solutions are considered in the Maldacena-Núñez background seem to indicate that only in those cases where supersymmetry is asymptotically restored can one really neglect the σ-model corrections.

One of the most interesting developments along these lines consists on trying to extract the string σ-model action directly from the gauge theory. In the latter one needs to consider a limit of the dilatation operator (interpreted as a certain spin-chain Hamiltonian) in which the angular momenta are large keeping λ/J^2 fixed. The resulting Hamiltonian has been matched

to the one arising from taking the same limit in the string σ-model [83, 166] in the case of two angular momenta and to second order in λ/J^2. If the dilatation operator could be computed at all loops, it would then be possible to reconstruct the whole σ-model from the gauge theory. The correspondence would then go beyond the matching of particular string states to particular SYM states.

In the branch of noncommutative theories we first investigated the toy model of a non-relativistic NC ϕ^4 theory at the quantum level. Apart from the UV/IR mixing, we showed that, despite their completely different treatment of time and space, the same rules of their relativistic counterparts apply: they are unitary except in electric backgrounds. However, they do not share the property that unitarity can be restored at the one-loop level by the addition of new asymptotic states (the χ-particles). This is not a big problem anyway as it would simply imply that there is no possibility to detect the undecoupled string modes by looking only at the field theory. In this case we would be talking about undecoupled excitations of a non-relativistic string theory [167].

Motivated by a proper quantization of the electric NC theories and, in general, of any field theory non-local in time, we have elaborated on the Hamiltonian formalism for such theories that was introduced in [29] and further developed in [30]. The formalism requires the introduction of an extra timelike coordinate with respect to which the Lagrangian is finally local. Its Hamiltonian formalism is essentially based on, apart from a certain Hamilton functional, a set of constraints that guarantee the equivalence with the original non-local theory in one dimension less. Within this framework we first provided the correct construction of the symmetry generators. It is remarkable that gauge transformations of the original Lagrangian end up as global transformation in the d+1 formalism. We have also shown that our Hamiltonian is indeed the generator of time translations, justifying its property of giving the energy of a given field configuration. In general, there are some contributions to the energy which give what one would have naively found by putting $*$-products in the expression for the energy of a commutative theory. However, there are some other contributions which are purely $(d+1)$-dimensional and they do not vanish in a generic configuration. We checked this explicitly for an electric NC $U(1)$ gauge theory (its energy functional is given in equation (VII.68)). We completed the our study of non-local theories with a BRST and field-antifield analysis within this formalism.

Finally, we have been able to study the string duals of NC field theories by means of D-branes wrapped in calibrated cycles in backgrounds with non-zero B_2-fields. The first lesson we learnt was a technical one: gauged supergravities do not help anymore in finding the supergravity solutions. The reason why is that the (in this cases) eleven-dimensional solution preserves a set of Killing spinors that do not survive the (in this case $SU(2)$) compactification. Remarkably, the resulting background does solve the gauged-supergravity equations of motion but it does not preserve supersymmetry. Thus one must choose between

- solving coupled *second order* partial differential equations in 8d supergravity (which is simpler than 11d supergravity),

- or solving coupled *first order* BPS equations directly in 11d supergravity.

We managed to use this second option and produce the supergravity dual of the NC deformation of the $\mathcal{N} = 2$ in 2+1 that we had previously studied. By reducing along the appropriate circle (avoiding the phenomenon of susy without susy) we obtained a rather complicated IIA solution in which the metric depends on two coordinates: one is the transverse direction to the D-branes inside the CY_3 and one outside it. A simple probe analysis showed that the moduli space exactly coincides with the commutative one, a signal that the UV/IR mixing leaves it unaffected.

By using an improved method based on a chain of T-dualities and the addition of a constant B_2-field in one of the steps, we were able to analyze the NC deformation of the Maldacena-Núñez background, which is dual to an $\mathcal{N} = 1$ $SU(N)$ gauge theory in 3+1 dimensions. Because this theory (the commutative one) shares so many features of QCD, the possibility of studying it non-perturbatively has received a lot of attention in the last years. We have analyzed a series of features of its NC version with the following results,

- there seems to be stronger effects of the UV/IR mixing. This is the only case we are aware of in which the background does not reduce to its commutative one in the deep IR, as can be seen by looking at coordinate invariants such as the curvature scalar. Neither the RR-forms completely reduce to the commutative ones.

- the computation of the Wilson line shows confinement at large distances with exactly the same string tension as in the commutative

version of the theory. The UV behavior does vary a lot, as expected in a region were the noncommutativity scale is perfectly visible.

- the computation of the $U(N)$ NC β-function shows that it coincides exactly with the $SU(N)$ commutative one. This is already to be expected from general perturbation theory considerations [34, 130]. Essentially, the $U(1)$ and the $SU(N)/Z_N$ factors inside the NC $U(N)$ run with the same coupling constant, and the divergent contributions to it come from planar diagrams (as Θ acts as a cut off for non-planar ones).

- the introduction of the NC scale Θ makes it possible to decouple the unwanted KK modes. This condition, together with negligibility of string loop corrections and the condition that the curvatures be small can be written, in units where $l_s = 1$,

$$\frac{e^{-3\Phi_0}}{\Theta^2} \ll N \ll e^{-\Phi_0} \ll \Theta\,. \qquad \text{(VIII.1)}$$

  Thus we see that the only way to decouple the KK modes is to let Θ be the largest scale of the problem. This shows that we cannot use NC deformations of backgrounds with decoupling problems if our aim is to end up with a realistic theory. However, this procedure could help to clarify the role of the KK modes in the future.

Note that the fact that the background does not tend to its commutative version in the deep IR does not seem to affect the deep IR behavior of the confining string nor the β-function. It would be interesting to think of a way of probing this difference in the supergravity side and then translating it into some field theory effect.

The future of the research lines studied here seems to strongly depend on our capability to understand string theory in backgrounds with RR fluxes, as these appear inevitably in the closed string description of D-branes. Progress along this direction would extend the validity of the stringy calculations in a fashion similar to the BMN correspondence. Indeed, not only the AdS/CFT duality would be much better understood, but also the more general gauge/string correspondence, and the latter would immediately improve our understanding of non-perturbative phenomena in QCD-like theories. It may well happen that when quantization in such backgrounds will be possible, all the closed string duals obtained so far will receive a renewed interest from field theorists. Finally, let us remark that all these considerations do not depend on whether string/M-theory is the final theory describing

nature (if it describes it at all!), a fact that makes all results obtained in this context more solid and that allows the gauge/string correspondence to constitute an almost independent field on its own.

APPENDIX

A Superconformal algebra, representations and BPS operators

It is of crucial importance to understand a few properties of the superconformal algebra and its representations. Let us start with recalling the commutation relations of the algebra. We will write them in a schematic form,

$$[M,P]=P\,,\qquad [M,K]=K\,,\qquad [M,M]=M\,,\qquad [M,D]=0$$
$$[D,K]=K\,,\qquad [D,P]=-P\,,\qquad [P,K]=M+D\,. \tag{A-1}$$

The first line just defines the tensorial behavior of the generators and the second one assigns conformal weights $[K]=-1$ and $[P]=+1$. Unitarity of the representation requires that all its states have conformal weight (eigenvalue under D) $\Delta\geq 0$, so by successive application of K to any definite scaling state we obtain one which is annihilated by all K's; such a state is called a *conformal primary state*.

The supersymmetric extension of the conformal algebra introduces Poincaré Q and conformal S fermionic generators whose (anti)commutation relations read

$$[D,S]=\frac{1}{2}S\,,\qquad [D,Q]=-\frac{1}{2}Q\,,\qquad \{Q,Q\}=P\,,\qquad \{S,S\}=K$$
$$[K,Q]=S\,,\qquad [P,S]=Q\,,\qquad \{Q,S\}=M+D+R\,. \tag{A-2}$$

Again, the first two relations imply that $[S]=-1/2$ and $[Q]=1/2$. Repeating the argument above, we can apply S to any state until we reach one annihilated by all the S's. Such a state is called *superconformal primary* if, in addition to this, it is not the case that this state can be written as Q acting on another state. This subtlety is important because the construction of a superconformal multiplet typically begins with the identification of such a superconformal primary, and by successive application of the rest of the generators, including the Q's. If our candidate to start this procedure was annihilated by all the S's but it was the Q of some other state, then we would not construct the true whole multiplet but just a part of it, since $Q^2=0$. To summarize:

$$\text{Superconformal primary state } |O\rangle \iff S_\alpha|O\rangle=0,\ |O\rangle\neq Q|O'\rangle$$

If the superconformal primary state we start with is not annihilated by any of the Q's then the multiplet constructed upon it is called a *long*

multiplet. Some special cases occur when the superconformal primary state is itself annihilated by some of the Q's, in which case its multiplet contains much less states than a long one. It can be proven that the only possibilities are that $1/n$ of the Q's annihilate it with $n = 2, 4, 8$. Such superconformal primary states are called $1/n$ *chiral primary states* and their corresponding supermultiplets are called $1/n$-BPS multiplets. The name BPS is justified by the fact that the superconformal primary states in these multiplets have the smallest possible conformal dimension among all states with the same remaining quantum numbers. It can be further proven that the only nonzero charges of chiral primary state can be R-symmetry charges. If we name their $SO(2)^3 \subset SO(6)$ Cartan charges by (J_1, J_2, J_3), then the chiral primaries saturate the BPS bound

$$\Delta \geq |J_1| + |J_2| + |J_3| \,. \tag{A-3}$$

Having three, two or one nonzero charges J_i turns the corresponding multiplet into a 1/8-, 1/4- or 1/2- BPS one. It is easy to intuitively understand how such bounds arise, for if a state is annihilated by all of the S's and at least one of the Q's then the $\{Q, S\} = M + D + R$ anticommutator automatically provides a relation among the conformal dimension and the R-symmetry charges. Being a purely algebraic relation, the conformal dimension of such states does not depend on the YM coupling, and it is therefore an exact non-renormalized statement. We remark that this property is automatically inherited by the descendants of the primary.

From states to operators.

We have avoided mentioning about operators in the discussion above. However, conformal theories are known to admit a 1-to-1 map between states in the radial quantization $|O\rangle$ and local operators $O(x)$, the map being given by $|O\rangle = \lim_{x\to 0} O(x)$.

It is then immediate to ask what are the operators that correspond to the *superconformal primary states*. A first intuitive argument to find them is that such operators must be build only out of scalar fields. The reason follows from observation of the supersymmetry transformation of the $\mathcal{N} = 4$ fields

$$[Q, \phi] \sim \lambda \,, \qquad \{Q, \lambda\} \sim F_2 + [\phi, \phi] \,, \qquad \{Q, \bar{\lambda}\} \sim D_\mu \phi \,, \qquad [Q, A_1] \sim \bar{\lambda}.$$

As we are looking for fields that are not the Q of anything then they cannot be fermions nor gauge fields, for they appear in the RHS of the

Q-transformation for scalars and fermions respectively. The same argument forces the scalars to be symmetrized, as $[\phi^i, \phi^j]$ appears in the Q-transformation of the fermions. The simplest superconformal primaries are then writable as

$$O_{1,n} = \mathrm{str}\left(\phi^{i_1}\phi^{i_n}\ldots\phi^{i_n}\right)\,, \tag{A-4}$$

where str stands for the symmetrized trace and the subscripts in $O_{p,n}$ indicate that the operator contains p traces and n scalars, so that the operator (A-4) is a single-trace operator. Multiple trace operators can be build by products of single-trace ones and they allow for the appearance of partially anti-symmetrized representations of SO(6). The operators $O_{p,n}$ exhaust the list of known superconformal primary operators.

Let us now look for *chiral primary* operators, *i.e.* we further require that the superconformal primary $O_{p,n}$ is annihilated by at least one of the Q's. The answer is less intuitive so we just give the results:[1]

- **1/2-BPS Operators.** All we need to do is to select the traceless combination of an operator $O_{p,n}$. This means that they all transform in the $[0, n, 0]$ Dynkin irrep of SO(6). In terms of complexified scalar fields

  $$X = \phi^1 + i\phi^2\,, \qquad Y = \phi^3 + i\phi^4\,, \qquad Z = \phi^5 + i\phi^6\,, \tag{A-5}$$

  the single and double trace 1/2-BPS operators are simply

  $$\begin{aligned} \mathcal{O}_{1,J} &= \mathrm{Tr}(\overbrace{X\ldots X}^{J}) \\ \mathcal{O}_{2,J} &= \sum_{p=0}^{J} a_p\,\mathrm{Tr}(\overbrace{X\ldots X}^{J-p})\mathrm{Tr}(\overbrace{X\ldots X}^{p})\,, \end{aligned} \tag{A-6}$$

  where in the double trace operator one needs to choose the coefficients a_p in order to make the operator transform in the $[0, J, 0]$ irrep.

- **1/4-BPS Operators.** In this case the 1/4 requirement implies that one must select the $[l, k, l]$ Dynkin irrep of SO(6) with $k + 2l$ being equal to the number of scalars n in the operator. For $l = 0$ we recover the 1/2 operators, and for $l \neq 0$ we clearly see that we need at least 2 traces. So 1/4-BPS operators are those $O_{p,n}$ with $p \geq 2$ that transform in $[l, k, l]$ Dynkin irreps with $k + 2l = n$.

[1] In what follows we will often use Dynkin labels instead of $SO(2)^3$ charges; the former are written in squared brackets $[p_1, p_2, p_3]$ and the latter in parenthesis (J_1, J_2, J_3). The relation between the two is that (J_1, J_2, J_3) corresponds to $[J_2 - J_3, J_1 - J_2, J_2 + J_3]$ for $J_1 \geq J_2 \geq J_3$.

- **1/8-BPS Operators.** In this case the 1/8 requirement implies that one must select the $[l, k, l+2m]$ Dynkin irrep of SO(6) with $k+2l+3m=n$. We now see that we need at least 3 traces.

We summarize this discussion in the following table, which we adapt from [168]. The table refers to the chiral primary representative of the irrep and it shows the number of Q's that annihilate it, and its R-quantum numbers in two formats, according to its charges under $SO(2)^3 \subset SO(6)$ or according to its Dynkin labels.

Operator type	$\#Q$	$SO(2)^3$ charges	Dynkin labels	dimension Δ
identity	16	$(0,0,0)$	$[0,0,0]$	0
1/2 BPS	8	$(J_1,0,0)$	$[0,k,0],\ \ k \geq 2$	k
1/4 BPS	4	(J_1,J_2,J_2)	$[\ell,k,\ell],\ \ \ell \geq 1$	$k+2\ell$
1/8 BPS	2	(J_1,J_2,J_3)	$[\ell,k,\ell+2m]$	$k+2\ell+3m,\ \ m \geq 1$
non-BPS	0	any	any	unprotected

Tab. .1: Characteristics of BPS and Non-BPS multiplets

B Generalization to strings with 3 independent angular momenta

In this appendix we describe more general string solutions which carry 3 independent angular momenta. The trick consists on winding the string differently in the 3 planes contained in $\mathbb{R}^6$, as proposed in [11]. We also redo the κ-symmetry analysis. As we know that they all become tensionless in the limit, we will start directly with the Lagrangian and κ-symmetry transformations suitable for tensionless strings. This will simplify the problem considerably.

Let us rewrite them metric (III.29) as

$$ds^2 = -dt^2 + d\theta^2 + \sin^2\theta d\phi^2 + \sum_{i=1}^{3} a_i^2 (d\alpha_i)^2 \,, \tag{B-7}$$

where $a_1 = \cos\theta$, $a_2 = \sin\theta\cos\phi$, $a_3 = \sin\theta\sin\phi$, and α_i are polar angles in three orthogonal planes. In order to avoid coordinate singularities, we will assume here that $a_1a_2a_3$ is non-zero, which corresponds to the assumption that the angular momentum two-form has maximal rank.

Let (τ,σ) be the worldsheet coordinates. In the gauge $t=\tau$ the phase-space form of the Lagrangian density for a tensionless string in the above background is

$$L = p_\theta\dot\theta + p_\phi\dot\phi + \sum_i p_i\dot\alpha_i - H - s\left(p_\theta\theta' + p_\phi\phi' + \sum_i p_i\alpha_i'\right) \,, \tag{B-8}$$

where s is the Lagrange multiplier for the string reparametrization constraint, and H is the Hamiltonian density

$$H = \sqrt{p_\theta^2 + \frac{p_\phi^2}{\sin\theta^2} + \sum_i \frac{p_i^2}{a_i^2}} \,. \tag{B-9}$$

Note that the α_i equation of motion is $\dot p_i = (sp_i)'$, from which it follows that the angular momentum, $J_i = \oint d\sigma\, p_i$, is a constant of motion.

We seek solutions for which θ and ϕ are constant and $p_\theta = p_\phi = 0$. This requires, in order to solve the corresponding equations of motion, that we set

$$a_i^2 = |p_i| \Big/ \sum_j |p_j| \,, \tag{B-10}$$

from which we see that

$$H = \sum_i |p_i| \,. \tag{B-11}$$

Given (B-10), the p_i equations of motion reduce to $\dot{\alpha}_i - s\alpha_i' = 1$, while the constraint imposed by s is $\sum_i p_i \alpha_i' = 0$. We may choose a gauge for which $s = 0$, in which case the equations above have the solution[2]

$$\alpha_i = t + m_i \sigma \,, \qquad p_i = J_i / 2\pi \,, \tag{B-12}$$

for integers m_i satisfying $\sum_i m_i J_i = 0$. Because each p_i has a definite sign, integration of (B-11) yields the total energy

$$E = |J_1| + |J_2| + |J_3| \,. \tag{B-13}$$

An argument analogous to that of section A shows that this energy saturates a BPS bound implied by the $PSU(2,2|4)$ supersymmetry algebra of $AdS_5 \times S^5$. Because of this, the rotating strings are supersymmetric. Since we have assumed that all J_i are non-zero, the fraction of supersymmetry preserved is 1/8, as we will confirm below. If only two J's are non-zero, then 1/4 supersymmetry is preserved. And if only one J is non-zero, then 1/2 supersymmetry is preserved.

Note that the strings with two equal angular momenta considered in section III.4 are a subclass of these more general ones; as soon as we the winding numbers m_i are different, one can have three different angular momenta. The novelty is that now it is not true anymore that the string lies in a two-plane at each moment; it describes a closed curve that spans (in general) the whole $\mathbb{R}^6$.

Let us now redo the κ-symmetry analysis of section III.4.4 for these new cases. The condition for a IIB superstring to be supersymmetric in the ultra-relativistic limit in which it becomes null is

$$p_M \, e^M{}_A \, \Gamma^A \, \epsilon = 0 \,, \tag{B-14}$$

where p_M is the ten-momentum and $e^M{}_A$ is the obvious orthonormal frame associated to the metric (B-7). The spinor (III.53) in our renamed coordinates is

$$\epsilon = e^{i\frac{t}{2}\tilde{\Gamma}} e^{i\frac{\theta}{2}\Gamma_{\phi 123}} e^{\frac{\phi}{2}\Gamma_{\theta\phi}} e^{\frac{i}{2}\left(\alpha_1 \Gamma_{\theta\phi 23} + \alpha_2 \, i\Gamma_{2\theta} + \alpha_3 \, i\Gamma_{3\phi}\right)} \epsilon_0 \,, \tag{B-15}$$

where recall that $\tilde{\Gamma}$ commutes with all other matrices in the problem and $\Gamma_i \equiv \Gamma_{\alpha_i}$. It is clear from this expression that the matrices occurring inside

[2] This is not the unique solution, but it is the one of relevance when one considers the tensionless string as a limit of the tensionful string.

the brackets in the last exponential generate, when acting on ϵ_0, rotations in each of the three orthogonal two-planes parametrized by α_i. Note that they all square to unity (so they have eigenvalues ± 1), they are mutually-commuting, and the product of any two of them yields the third one, up to a sign.

For the rotating string solutions of the previous section, the supersymmetry condition (B-14) reduces to

$$a_i \, \Gamma_{ti} \, \epsilon = \epsilon \,, \tag{B-16}$$

where we have assumed, for definiteness, that all J_i are positive. If the angular momentum two-form has maximum rank then all a_i are non-zero, and equation (B-16) is equivalent to

$$i\Gamma_{2\theta} \, \epsilon_0 = \epsilon_0 \,, \qquad i\Gamma_{3\phi} \, \epsilon_0 = \epsilon_0 \,, \qquad \Gamma_{t1} \, \epsilon_0 = \epsilon_0 \,. \tag{B-17}$$

Let us first show that (B-17) implies (B-16). We see from the first two conditions in (B-17) that ϵ_0 is an eigen-spinor of the last exponential in equation (B-15), so this exponential cancels on both sides of (B-16). The first exponential also cancels because $\tilde{\Gamma}$ commutes with all other matrices in equation (B-16), so the latter may be rewritten as

$$a_i \, \Gamma_{ti} \, \epsilon_0 = e^{-\frac{\phi}{2}\Gamma_{\theta\phi}} e^{i\theta\Gamma_{\phi 123}} e^{\frac{\phi}{2}\Gamma_{\theta\phi}} \, \epsilon_0 = \left(a_1 + a_2 \, i\Gamma_{\phi 123} + a_3 \, i\Gamma_{123\theta} \right) \epsilon_0 \,, \tag{B-18}$$

which is identically satisfied by virtue of (B-17).

With some further algebra, it can also be shown that (B-16) implies (B-17), but we will not do this here. Instead, we note that, being mutually-commuting projections, the conditions (B-17) imply that the fraction of preserved supersymmetry is 1/8, as expected for a string carrying three independent angular momenta. We also recall that, as observed above, they imply that ϵ_0 is invariant (up to a phase) under rotations in the α_i-planes associated to the non-zero components of the angular momentum two-form.

C Conventions for the Maldacena-Núñez background

In this appendix we collect the conventions and definitions necessary to read the Maldacena-Núñez background in (VI.7)-(VI.9), referring the reader to the original references for a careful derivation. There are two functions of the radial variable ρ, which are given by

$$\begin{aligned} e^{2g(\rho)} &= \rho\coth 2\rho - \frac{\rho^2}{\sinh^2 2\rho} - \frac{1}{4}\,, \\ a(\rho) &= \frac{2\rho}{\sinh 2\rho}\,. \end{aligned} \tag{C-2}$$

The $SU(2)$ gauge-field A is parametrized by

$$A = \frac{1}{2}\left[\sigma^1 a(\rho)d\theta + \sigma^2 a(\rho)\sin\theta d\phi + \sigma^3\cos\theta d\phi\right]\,, \tag{C-3}$$

where

$$A = A^a\frac{\sigma^a}{2}\,, \qquad F = F^a\frac{\sigma^a}{2}\,, \tag{C-4}$$

and $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc}A^b_\mu A^c_\nu$ should be understood. Finally, the $SU(2)$ left-invariant one forms parametrizing the transverse S^3 are

$$\begin{aligned} w^1 + iw^2 &= e^{-i\psi}\left(d\theta_1 + i\sin d\phi_1\right)\,, \\ w^3 &= d\psi + cos\theta_1 d\phi_1. \end{aligned} \tag{C-5}$$

D $U(1)$ commutative Maxwell theory in d+1 dimensions

Our $d+1$ formalism can also be used for describing ordinary local theories. As an example of this, we will show how the $U(1)$ commutative (and therefore local) Maxwell theory is formulated using the d+1 dimensional canonical formalism developed for non-local theories and see how it is reduced to the standard canonical formalism in d dimensions.

The canonical $d+1$ system is defined by the Hamiltonian (VII.14) and two constraints, (VII.16) and (VII.20). The Hamiltonian is

$$H = \int d^d x \, [\Pi^\nu(t,x)\partial_{x^0}\mathcal{A}_\nu(t,x) - \delta(x^0)\mathcal{L}(t,x)\,], \tag{D-2}$$

where

$$\mathcal{L}(t,x) = -\frac{1}{4}\,\mathcal{F}_{\mu\nu}(t,x)\mathcal{F}^{\mu\nu}(t,x), \tag{D-3}$$

$$\mathcal{F}_{\mu\nu}(t,x) = \partial_\mu\mathcal{A}_\nu(t,x) - \partial_\nu\mathcal{A}_\mu(t,x). \tag{D-4}$$

The momentum constraint (VII.16) is

$$\varphi^\nu(t,x) = \Pi^\nu(t,x) + \int dy\ \chi(x^0,-y^0)\ \mathcal{F}^{\mu\nu}(t,y)\ \partial^y_\mu\delta(x-y) \tag{D-5}$$

$$= \Pi^\nu(t,x) + \delta(x^0)\mathcal{F}^{0\nu}(t,x) \approx 0\,, \tag{D-6}$$

and the constraint (VII.20) is

$$\tilde{\varphi}^\nu(t,x) = \partial_\mu\mathcal{F}^{\mu\nu}(t,x) \approx 0. \tag{D-7}$$

The generator of the $U(1)$ transformation is given, using (VII.41), by

$$G[\Lambda] = \int dx[\ \Pi^\mu\partial_\mu\Lambda\ ]. \tag{D-8}$$

It can be expressed as a linear combination of the constraints,

$$G[\Lambda] = \int dx\ \Lambda\left[-(\partial_\mu\varphi^\mu) - \delta(x^0)\tilde{\varphi}^0\right]. \tag{D-9}$$

The Hamiltonian is expressed using the constraints and the $U(1)$ generator as

$$H = G[\mathcal{A}_0]\ +\int dx\ \varphi^i\ \mathcal{F}_{0i}\ +\int dx\ \delta(x^0)\ \{\frac{1}{2}{\mathcal{F}_{0i}}^2 + \frac{1}{4}{\mathcal{F}_{ij}}^2\}. \tag{D-10}$$

The Hamiltonian (D-10) as well as the constraints (D-6) and (D-7) contain no time (t) derivative and they are functions of the canonical pairs $(\mathcal{A}_\mu(t,x), \mathit{\Pi}^\mu(t,x))$. They are conserved since the Maxwell Lagrangian in d dimensions has time translation invariance. The $U(1)$ generator is also conserved, even without using the constraints,

$$\frac{d}{dt}G[\Lambda] \;=\; \{G[\Lambda], H\} \;+\; \frac{\partial}{\partial t}G[\Lambda] \;=\; 0\,, \tag{D-11}$$

for $\Lambda(t,x)$ satisfying (VII.43),

$$\dot{\Lambda} = \partial_{x^0}\Lambda\,, \tag{D-12}$$

in agreement with (VII.46). The condition on Λ implies that the $U(1)$ transformations in the d+1 dimensional canonical formulation are not gauge but rigid ones. We will see below how the gauge transformations are recovered when reducing back to the original d dimensional formalism.

In cases where our Lagrangians are local or higher derivative ones we can expand our fields as in (VII.22) to reduce back to d dimensions,

$$\mathcal{A}_\mu(t,x) \equiv \sum_{m=0}^{\infty} \; e_m(x^0)\; A_\mu^{(m)}(t,\mathbf{x})\,, \qquad \mathit{\Pi}^\mu(t,x) \equiv \sum_{m=0}^{\infty} \; e^m(x^0)\; \mathit{\Pi}^\mu_{(m)}(t,\mathbf{x}), \tag{D-13}$$

The fields $(A_\mu^{(m)}(t,\mathbf{x}), \mathit{\Pi}^\mu_{(m)}(t,\mathbf{x}))$ are the new symplectic coordinates in d dimensions. In terms of them, the constraint (D-6) can be expressed as

$$\varphi^\mu(t,x) \;=\; \sum_{m=0}^{\infty} e^m(x^0)\varphi^\mu_{(m)}(t,\mathbf{x}), \tag{D-14}$$

$$\varphi^0_{(m)}(t,\mathbf{x}) \;=\; \mathit{\Pi}^0_{(m)}(t,\mathbf{x}) \;=\; 0\,, \qquad (m \ge 0), \tag{D-15}$$

$$\varphi^i_{(0)}(t,\mathbf{x}) \;=\; \mathit{\Pi}^i_{(0)}(t,\mathbf{x}) - (\mathcal{A}_i^{(1)}(t,\mathbf{x}) - \partial_i\mathcal{A}_0^{(0)}(t,\mathbf{x})) \;=\; 0, \tag{D-16}$$

$$\varphi^i_{(m)}(t,\mathbf{x}) \;=\; \mathit{\Pi}^i_{(m)}(t,\mathbf{x}) \;=\; 0\,, \qquad (m \ge 1). \tag{D-17}$$

and the constraint (D-7) as

$$\tilde{\varphi}^\mu(t,x) \;=\; \sum_{m=0}^{\infty} e_m(x^0)\tilde{\varphi}^{\mu(m)}(t,\mathbf{x}), \tag{D-18}$$

$$\begin{aligned}\tilde{\varphi}^{i(m)}(t,\mathbf{x}) \;&=\; \partial_j\left(\partial_j\mathcal{A}_i^{(m)}(t,\mathbf{x}) - \partial_i\mathcal{A}_j^{(m)}(t,\mathbf{x})\right)\\ &-\; \left(\mathcal{A}_i^{(m+2)}(t,\mathbf{x}) - \partial_i\mathcal{A}_0^{(m+1)}(t,\mathbf{x})\right) = 0, (m \ge 0)\end{aligned} \tag{D-19}$$

$$\tilde{\varphi}^{0(m)}(t,\mathbf{x}) \;=\; \partial_i\left(\mathcal{A}_i^{(m+1)}(t,\mathbf{x}) - \partial_i\mathcal{A}_0^{(m)}(t,\mathbf{x})\right) = 0,\ (m \ge 0). \tag{D-20}$$

Due to the identities

$$\tilde{\varphi}^{0(m+1)}(t,\mathbf{x}) \;=\; \partial_i\tilde{\varphi}^{i(m)}(t,\mathbf{x})\,, \qquad (m \geq 0)\,, \tag{D-21}$$

the only independent constraint of (D-20) is the $m = 0$ case. It can be expressed, using (D-16), as the gauss law constraint,

$$\tilde{\varphi}^{0(0)}(t,\mathbf{x}) \;=\; \partial_i \mathit{\Pi}^i_{(0)}(t,\mathbf{x}) \;=\; 0. \tag{D-22}$$

Following the Dirac's standard procedure for dealing with constraints [169], we classify them and eliminate the second class ones. The constraints (D-17) with $(m \geq 2)$ are paired with the constraints (D-19) with $(m \geq 0)$ to form second class sets. They are used to eliminate the canonical pairs $(\mathcal{A}_i^{(m)}(t,\mathbf{x}), \mathit{\Pi}^i_{(m)}(t,\mathbf{x})), (m \geq 2)$,

$$\begin{aligned} \mathcal{A}_i^{(m)}(t,\mathbf{x}) \;&=\; \partial_j(\partial_j\mathcal{A}_i^{(m-2)}(t,\mathbf{x}) - \partial_i\mathcal{A}_j^{(m-2)}(t,\mathbf{x})) + \partial_i\mathcal{A}_0^{(m-1)}(t,\mathbf{x}),\\ \mathit{\Pi}^i_{(m)}(t,\mathbf{x})) \;&=\; 0, \qquad\qquad (m \geq 2). \end{aligned} \tag{D-23}$$

The constraints (D-17) with $(m = 1)$ and (D-16) are paired to a second class set and can be used to eliminate $(\mathcal{A}_i^{(1)}(t,\mathbf{x}), \mathit{\Pi}^i_{(1)}(t,\mathbf{x}))$,

$$\mathcal{A}_i^{(1)}(t,\mathbf{x}) \;=\; \mathit{\Pi}^i_{(0)}(t,\mathbf{x}) + \partial_i\mathcal{A}_0^{(0)}(t,\mathbf{x}), \tag{D-24}$$

$$\mathit{\Pi}^i_{(1)}(t,\mathbf{x}) \;=\; 0. \tag{D-25}$$

After eliminating the canonical pairs $(\mathcal{A}_i^{(m)}(t,\mathbf{x}), \mathit{\Pi}^i_{(m)}(t,\mathbf{x})), (m \geq 1)$ using the second class constraints, the system is described in terms of the canonical pairs $(\mathcal{A}_i^{(0)}(t,\mathbf{x}), \mathit{\Pi}^i_{(0)}(t,\mathbf{x}))$ and $(\mathcal{A}_0^{(m)}(t,\mathbf{x}), \mathit{\Pi}^0_{(m)}(t,\mathbf{x})), (m \geq 0)$. The Dirac brackets among them remain the same as the Poisson brackets. Remember that the d dimensional fields can be read from (VII.7) as

$$A_\mu(t,\mathbf{x}) = \mathcal{A}_\mu(t,0,\mathbf{x}) \;=\; \mathcal{A}_\mu^{(0)}(t,\mathbf{x})\,, \qquad \Pi^\mu(t,\mathbf{x}) = \mathit{\Pi}^\mu_{(0)}(t,\mathbf{x}). \tag{D-26}$$

The remaining constraints are (D-22) and (D-15),

$$\partial_i\mathit{\Pi}^i_{(0)}(t,\mathbf{x}) \;=\; 0, \qquad\qquad \mathit{\Pi}^0_{(m)}(t,\mathbf{x}) \;=\; 0\,, \qquad (m \geq 0)\,. \tag{D-27}$$

They are first class constraints. The Hamiltonian (D-10) in the reduced variables is

$$H(t) \;=\; \int d\mathbf{x}\left[\,\sum_{m=0}^{\infty}\;\mathcal{A}_0^{(m+1)}(t,\mathbf{x})\mathit{\Pi}^0_{(m)}(t,\mathbf{x}) \;-\; \mathcal{A}_0^{(0)}(t,\mathbf{x})(\partial_i\mathit{\Pi}^i_{(0)}(t,\mathbf{x}))\right. \tag{D-28}$$

$$\left. +\;\frac{1}{2}(\mathit{\Pi}^i_{(0)}(t,\mathbf{x}))^2 \;+\; \frac{1}{4}(\partial_j\mathcal{A}_i^{(0)}(t,\mathbf{x}) - \partial_i\mathcal{A}_j^{(0)}(t,\mathbf{x}))^2\right]. \tag{D-29}$$

The $U(1)$ generator (D-9) is

$$G[\Lambda] = \int d\mathbf{x} \left[\sum_{m=0}^{\infty} \Lambda^{(m+1)}(t,\mathbf{x})\mathit{\Pi}^0_{(m)}(t,\mathbf{x}) - \Lambda^{(0)}(t,\mathbf{x})(\partial_i \mathit{\Pi}^i_{(0)}(t,\mathbf{x}))\right], \tag{D-30}$$

where

$$\Lambda(t,\lambda) = \sum_{m=0}^{\infty} \Lambda^{(m)}(t,\mathbf{x})e_m(x^0)\,, \qquad \text{and} \qquad \dot{\Lambda}^{(m)}(t,\mathbf{x}) = \Lambda^{(m+1)}(t,\mathbf{x}). \tag{D-31}$$

The first class constraints $\mathit{\Pi}^0_{(m)}(t,\mathbf{x}) = 0, (m \geq 0)$ in (D-27) mean that $\mathcal{A}_0^{(m)}(t,\mathbf{x})$, $(m \geq 0)$ are the gauge degrees of freedom and we can assign to them any function of $\mathbf{x}$ for all values of m at given time $t = t_0$. It is equivalent to saying that we can assign any function of time to $\mathcal{A}_0^{(0)}(t,\mathbf{x})$ for all value of t, due to the equation of motion $\dot{\mathcal{A}}_0^{(m)}(t,\mathbf{x}) = \mathcal{A}_0^{(m+1)}(t,\mathbf{x})$. In this way we can understand that the Hamiltonian (D-29) is equivalent to the standard form of the canonical Hamiltonian of the Maxwell theory,

$$H(t) = \int d\mathbf{x} \Big[\dot{A}_0(t,\mathbf{x})\Pi^0(t,\mathbf{x}) - A_0(t,\mathbf{x})(\partial_i \Pi^i(t,\mathbf{x})) \tag{D-32}$$

$$+ \frac{1}{2}(\Pi^i(t,\mathbf{x}))^2 + \frac{1}{4}(\partial_j A_i(t,\mathbf{x}) - \partial_i A_j(t,\mathbf{x}))^2 \Big]\,, \tag{D-33}$$

in which $A_0(t,\mathbf{x})$ is an arbitrary function of time. In the same manner the $U(1)$ generator (D-30) is

$$G[\Lambda] = \int d\mathbf{x} \left[\dot{\lambda}(t,\mathbf{x})\Pi^0(t,\mathbf{x}) - \lambda(t,\mathbf{x})(\partial_i \Pi^i(t,\mathbf{x}))\right], \tag{D-34}$$

in which the gauge parameter function $\lambda(t,\mathbf{x}) \equiv \Lambda^{(0)}(t,\mathbf{x})$ is regarded as any function of time.

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